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RANDOM WALK ON A SPHERE AND ON A RIEMANNIAN MANIFOLD

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CONTENTS

A random walk on a sphere consists of a chain of random steps for which all directions from the starting point are equally probable, while the length α of the step is either fixed or subject to a given probability distribution $p(\alpha)$. The discussion allows the fixed length α or given distribution $p(\alpha)$, to vary from one step of the chain to another. A simple formal solution is obtained for the distribution of the moving point after any random walk; the simplicity depends on the fact that the individual steps commute and therefore have common eigenfunctions. Results are derived on the convergence of the eigenfunction expansion and on the asymptotic behaviour after a large number of random steps. The limiting case of diffusion is discussed in some detail and compared with the distribution propounded by Fisher (1953).

The corresponding problem of random walk on a general Riemannian manifold is also attacked. It is shown that commutability does not hold in general, but that it does hold in completely harmonic spaces and in some others. In commutative spaces, complete analogy with the method employed for a sphere is found.

1. Statement of the problems

Suppose that a probability distribution is given for the initial position $\mathbf{r}_0$ of a point on the surface of a sphere. The point receives a sequence of displacements, taking it to other points $\mathbf{r}_1, \mathbf{r}_2, \ldots$ on the sphere. Each displacement is of angular distance α, but is randomly directed from its starting point. We wish to find the probability distribution ρ_t of $\mathbf{r}_t$, and in particular the asymptotic behaviour of that distribution for large values of t.

The initial distribution may be one of absolute certainty that $\mathbf{r}_0$ is at Z, the 'north pole' of the sphere. The problem in this case is contained in the general problem, either as a particular case or as a limiting case: and the solution of the particular problem will give by superposition the solution of the general problem.

The problem may be generalized in three ways.

(i) The initial distribution is given by a probability density $\rho_0(\mathbf{r})$, the probability that $\mathbf{r}_0$ is in a solid angle $\mathrm{d}\omega$ about $\mathbf{r}$ being $\rho_0(\mathbf{r})\,\mathrm{d}\omega$. This density satisfies the conditions†

$$\rho_0(\mathbf{r}) \geqslant 0, \quad \iint \rho_0(\mathbf{r})\,\mathrm{d}\omega = 1. \tag{1}$$

† These conditions we state here for the case of finite density ρ_0. They require restatement for singular distributions.

The first extension is to abandon conditions (1) and their analogues for singular distributions. The distribution may then be called a charge distribution, the net total charge having any value, including zero. It is natural to assume that the total charge (without regard to sign) is finite,

$$Q_0 = \iint |\rho_0(\mathbf{r})| \, d\omega < \infty, \tag{2}$$

which, like (1), requires restatement for singular distributions.

This 'charge' generalization leaves quite unaffected the formal process of solution given in §§ 2, 3: indeed, the generalization is forced on to us by the method. As long as we retain the condition (2), we can write

$$\rho_0 = \rho_0' - \rho_0'', \quad |\rho_0| = \rho_0' + \rho_0'', \tag{3}$$

so that ρ_0', ρ_0'' are distributions everywhere non-negative: the corresponding total charges

$$Q_0' = \iint \rho_0' \, d\omega, \quad Q_0'' = \iint \rho_0'' \, d\omega$$

are both finite, since
$$Q_0 = Q_0' + Q_0'',$$

and so ρ_0', ρ_0'' differ from probability distributions only by the constant factors Q_0', Q_0''. Thus a random walk of a charge distribution is a linear combination of two random walks of probability distributions, so that the generalization—subject to (2)—cannot produce any substantial difference in the theory. Evidently

$$\begin{aligned} Q_1 = \iint |\rho_1| \, d\omega &= \iint |\rho_1' - \rho_1''| \, d\omega \\ &\leqslant \iint (\rho_1' + \rho_1'') \, d\omega = \iint \rho_1' \, d\omega + \iint \rho_1'' \, d\omega \\ &= \iint \rho_0' \, d\omega + \iint \rho_0'' \, d\omega = Q_0, \end{aligned}$$

so that $Q_t = \iint |\rho_t| \, d\omega$ decreases as t increases.

(ii) The second generalization allows the steps to be of different length: the displacement from $\mathbf{r}_{t-1}$ to $\mathbf{r}_t$ is randomly directed from $\mathbf{r}_{t-1}$ but is of angular distance α_t which may vary with t. This produces little formal change: but asymptotic properties† are affected if $\sin \alpha_t$ decays away too fast.

(iii) The third generalization replaces the step, random in direction but of fixed angular distance α, by a step still random in direction but of an angular magnitude α which is itself statistically distributed according to some law. In such a distribution let $p(\alpha)$ be the chance that the displacement is of angular distance $\leqslant \alpha$. Then $p(\alpha)$ is defined and monotonically increasing in the range $0 \leqslant \alpha \leqslant \pi$, and

$$0 \leqslant p(0) \leqslant p(\pi) = 1.$$

This generalization again produces little formal change: but asymptotic properties, and the analytic character of the solution for finite values of t, may be affected if the distribution of α shows too much concentration on small values of α or on values of α near to π.

† We shall use 'asymptotic' in relation to properties as $t \to \infty$. 'Convergence', on the other hand, will refer to the expansions in eigenfunctions.

The two generalizations may be combined, i.e. we may allow the statistical distribution $p_t(\alpha)$ of the angular magnitude α of the tth step to vary with t.

We may include as a limiting case a Brownian motion (Perrin 1925; Klein 1952): here the steps are infinitesimal, but their number is correspondingly increased. The difficulties (connected with asymptotic properties and convergence) at which we have hinted disappear in this case. The distribution obtained in this manner is not the same as one proposed by Fisher (1953) for dispersion on a sphere, but a detailed comparison reveals that the difference is small and, in practice, unimportant. This is fortunate, since the Brownian, or diffusion, distribution, while it may be theoretically correct, is analytically cumbersome, whereas Fisher's distribution is ideally suited for a statistical study.

All these problems are discussed in §§2, 3 and 4 below. The extension of this work to the case of a hypersphere of any number of dimensions would present no difficulty. However, in §5, we attempt the generalization to Riemannian manifolds. There is exact analogy if the space is of a type which we call commutative; that is, if any two random steps of any length commute. In particular, we show that a completely harmonic space is commutative and that the product of two commutative spaces is commutative. Commutative spaces are not necessarily harmonic and, in fact, we give an example of a closed three-dimensional commutative manifold which is not of constant curvature (and is not, therefore, harmonic; see Copson & Ruse 1940). It can be proved, however, that all two-dimensional commutative manifolds are of constant curvature.

2. The eigenfunctions and eigenvalues: the formal solution for a sphere

We take as basic the original problem with the first extension, and to avoid complexities of notation we suppose that the initial distribution is one of finite and continuous density $\rho_0(\mathbf{r}_0)$. Let $\rho_t(\mathbf{r}_t)$ be the distribution density after t steps. Then the law of formation is

$$\rho_t(\mathbf{r}) = \operatorname*{average}_{\widehat{\mathbf{r}'\mathbf{r}}=\alpha} \rho_{t-1}(\mathbf{r}'). \tag{4}$$

For the points $\mathbf{r}'$ appearing in this average are precisely those from which $\mathbf{r}$ can be reached in one further step. From symmetry, (4) holds apart from a possible factor independent of $\mathbf{r}$, and that factor must be unity to give conservation of total (net) 'charge' in the case in which ρ_{t-1} is a uniform distribution.

Equation (4) makes ρ_t a linear functional of ρ_{t-1}, and hence we have the principle of superposition: if

$$\rho_{t-1} = \sum_{(n)} \rho_{t-1,n},$$

then

$$\rho_t = \sum_{(n)} \rho_{t,n},$$

where $\rho_{t,n}$ is determined from $\rho_{t-1,n}$ by the same law (4). We look therefore for the characteristic values λ and characteristic functions $\phi_\lambda(\mathbf{r})$ of the problem, i.e. solutions of the equation

$$\operatorname*{average}_{\widehat{\mathbf{r}'\mathbf{r}}=\alpha} \phi_\lambda(\mathbf{r}') = \lambda\phi_\lambda(\mathbf{r}). \tag{5}$$

The solutions are immediately to hand: if $S_n(\mathbf{r})$ is a surface spherical harmonic of order n then

$$\operatorname*{average}_{\widehat{\mathbf{r}'\mathbf{r}}=\alpha} S_n(\mathbf{r}') = P_n(\cos\alpha)\, S_n(\mathbf{r}). \tag{6}$$

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To see this, take $\mathbf{r}$ as pole of spherical co-ordinates (θ, ψ) for $\mathbf{r}'$. Then $S_n(\mathbf{r}')$ is a linear combination of the zonal harmonic $P_n(\cos\theta)$ and the tesseral harmonics

$$P_n^{(m)}(\cos\theta)\cos m\psi, \quad P_n^{(m)}(\cos\theta)\sin m\psi.$$

The zonal harmonic satisfies (6), each side having the value $P_n(\cos\alpha)$: and a tesseral harmonic satisfies it, each side vanishing. Hence the general S_n also satisfies (6).

The characteristic values are therefore the numbers

$$\lambda_n = P_n(\cos\alpha), \tag{7}$$

and the characteristic functions corresponding to λ_n are all† the surface harmonics S_n of order n. Since the surface harmonics form a complete system there are no other characteristic values.

The characteristic functions do not depend† on the value of α. In consequence the same characteristic functions S_n arise in the third extension of the problem, but with characteristic values

$$\lambda_n = \int_0^\pi P_n(\cos\alpha)\,\mathrm{d}p(\alpha). \tag{8}$$

And the second extension also leaves the characteristic functions unchanged,† although the λ_n will then vary from step to step.

To derive the formal solution, expand $\rho_0(\mathbf{r})$ in a series of spherical harmonics,

$$\rho_0(\mathbf{r}) = \Sigma S_n(\mathbf{r}). \tag{9}$$

The series may be developed from the distribution by standard methods (cf. equation (28)): it need not converge in the strict sense, but from it the distribution may be determined uniquely. This is true even for singular distributions of finite total charge Q_0. If the distribution is one of finite continuous density ρ_0, then the series is certainly summable by the method of Abel and Borel.

For a problem without the second extension we get

$$\rho_1(\mathbf{r}) = \Sigma\lambda_n S_n(\mathbf{r}),$$

and in general

$$\rho_t(\mathbf{r}) = \Sigma\lambda_n^t S_n(\mathbf{r}), \tag{10}$$

with the same harmonic S_n throughout. For a problem with the second extension‡ we get

$$\rho_t(\mathbf{r}) = \Sigma\Lambda_{nt} S_n(\mathbf{r}), \tag{11}$$

† For a given n, there are $(2n+1)$ linearly independent harmonics S_n of order n. Thus the eigenvalue problem is inherently degenerate, λ_n being a $(2n+1)$-fold eigenvalue. *In general* there are no further degeneracies, the numbers $\lambda_0 = 1$, λ_1, λ_2, ... being distinct: the *only* characteristic functions are then the S_n of various orders. But it may happen that $\lambda_m = \lambda_n$ for a particular $p(\alpha)$ and two particular integers m, n. If so, there will be extra characteristic functions of the form $S_m + S_n$. In this way '*the* characteristic functions' do depend on the value of α in (7), or on the $p(\alpha)$ in (8); but there is no harm in ignoring such accidental degeneracies.

‡ The distribution after two steps of length α, β from the initial distribution (17) is given by a result of Dougall (1919)

$$\sum_0^\infty \frac{(2n+1)}{4\pi} P_n(\cos\alpha)\,P_n(\cos\beta)\,P_n(\cos\theta)$$
$$= \frac{1}{(2\pi)^2}\left[\sin\tfrac{1}{2}(\alpha+\beta+\theta)\,\sin\tfrac{1}{2}(\alpha+\beta-\theta)\,\sin\tfrac{1}{2}(\theta+\alpha-\beta)\,\sin\tfrac{1}{2}(\theta-\alpha+\beta)\right]^{-\frac{1}{2}},$$

if a spherical triangle can be drawn with sides α, β, θ, and zero otherwise. Whence, by the addition theorem for Legendre polynomials, the distribution after three steps can be expressed as a complete elliptic integral of the first kind. That the harmonic series for any number of steps is zero in inaccessible regions can be proved by induction.

with
$$\Lambda_{nt} = \prod_{k=1}^{t} \lambda_{nk}, \tag{12}$$
and, with the third extension as well,
$$\lambda_{nk} = \int_0^\pi P_n(\cos\alpha)\, \mathrm{d}p_k(\alpha). \tag{13}$$

For the Brownian problem (diffusion) we consider a large number N of steps, each of small angular distance α. We have
$$P_n(1) = 1, \quad P_n'(1) = \tfrac{1}{2}n(n+1),$$
$$\lambda_n = P_n(\cos\alpha) \approx P_n(1-\tfrac{1}{2}\alpha^2)$$
$$\approx 1-\tfrac{1}{4}n(n+1)\,\alpha^2 \approx \mathrm{e}^{-\frac{1}{4}n(n+1)\alpha^2}.$$
Hence
$$\rho_N = \Sigma\, \mathrm{e}^{-\frac{1}{4}n(n+1)V} S_n, \tag{14}$$
where V is the variance of the corresponding plane motion,
$$V = \Sigma\alpha^2. \tag{15}$$
For *equal* infinitesimal steps, we should have $V = N\alpha^2$. For a Brownian problem with the third extension we should get
$$V = \sum_{k=1}^{N} \int_0^\pi \alpha^2\, \mathrm{d}p_k(\alpha), \tag{16}$$
although these distinctions have no real significance for a genuine diffusion.

From an initial distribution of absolute certainty that the point is at the north pole, which has the (formal) expansion
$$\rho_0 = \Sigma \frac{2n+1}{4\pi} P_n(\cos\theta), \tag{17}$$
θ being the angular distance from the north pole, a diffusion of total variance V produces the final distribution
$$\rho = \Sigma \frac{2n+1}{4\pi} \mathrm{e}^{-\frac{1}{4}n(n+1)V} P_n(\cos\theta). \tag{18}$$
This would appear to be the correct analogue on the sphere of a symmetrical Gaussian distribution in the plane.

Fisher (1953) has studied the distribution
$$\mathrm{d}p(\alpha) = \frac{\kappa}{2\sinh\kappa} \mathrm{e}^{\kappa\cos\alpha} \sin\alpha\, \mathrm{d}\alpha. \tag{19}$$
If this be regarded as a single step in a random walk, it gives (Watson 1944, pp. 50, 77, 79)
$$\lambda_n = I_{n+\frac{1}{2}}(\kappa)/I_{\frac{1}{2}}(\kappa), \tag{20}$$
$$\lambda_{n+1} = \lambda_{n-1} - \frac{2n+1}{\kappa} \lambda_n, \tag{21}$$
$$\lambda_1 = \coth\kappa - 1/\kappa. \tag{22}$$
If κ is large, the step is small (i.e. the probability is large that α is small) and the variance is
$$\int \alpha^2 \mathrm{d}p(\alpha) \approx 2/\kappa.$$

Hence a large number N of such steps gives (14) with $V = 2N/\kappa$. In fact it can be shown directly that, for large κ,
$$\frac{I_{n+\frac{1}{2}}(\kappa)}{I_{\frac{1}{2}}(\kappa)} \approx 1 - \frac{n(n+1)}{2\kappa},$$
whence
$$\Lambda_{n,N} = \lambda_n^N \approx e^{-n(n+1)N/2\kappa}.$$

But it is of more interest to compare a *single* step of Fisher's type with a diffusion (14), (18). The exact analogue of (19) for diffusion is
$$dp'(\alpha) = \left[\sum_{n=0}^{\infty} (n+\tfrac{1}{2})\, e^{-\frac{1}{4}n(n+1)V} P_n(\cos\alpha)\right] \sin\alpha\, d\alpha, \tag{23}$$
and it gives
$$\lambda'_n = e^{-\frac{1}{4}n(n+1)V} = (\lambda'_1)^{\frac{1}{2}n(n+1)}. \tag{24}$$

We can attempt the comparison referred to by comparing the sequence $\{\lambda_n\}$ with the sequence $\{\lambda'_n\}$. We may normalize the comparison by imposing the condition
$$\lambda_1 = \lambda'_1, \tag{25}$$
so that the first harmonics are damped down in the same way. It then emerges that, while in each case the higher harmonics are more heavily damped than the lower (i.e. λ_n, λ'_n both decrease as n increases), this effect is more severe in diffusion than in Fisher's distribution. For example, for small κ we can† show that
$$\lambda_n \approx \frac{\kappa^n}{1.3.5\ldots(2n+1)},$$
so that λ'_n is an infinitesimal of higher order than λ_n as $\kappa \to 0$. Again, for large κ we can ignore the difference between $\coth\kappa$ and unity so that, using (21) and (22),
$$\lambda_1 \approx 1 - \frac{1}{\kappa}, \quad \lambda_2 \approx 1 - \frac{3}{\kappa}\left(1 - \frac{1}{\kappa}\right) = \lambda_1^3 + O(\kappa^{-3}),$$
and, by induction,
$$\lambda_n = \lambda_1^{\frac{1}{2}n(n+1)} + O(\kappa^{-3}),$$
the error term being positive; cf. equation (24).

Thus Fisher's distribution (19) imitates the diffusion distribution (23) very accurately for large κ (small V). Since both tend to the uniform distribution as $\kappa \to 0$ ($V \to \infty$) we may surmise that the discrepancy between them is never very great. This is confirmed by numerical computations. We may consider a distribution of Fisher's type $p(\alpha, \kappa)$ to fit a distribution of the Brownian type $p'(\alpha, V)$ best when the maximum difference between them

†
$$\lambda_n = \frac{\kappa}{2\sinh\kappa}\int_{-1}^{1} e^{\kappa\mu} P_n(\mu)\, d\mu$$
$$\Sigma\lambda_n x^n \equiv \frac{\kappa}{2\sinh\kappa}\int_{-1}^{1} \frac{e^{\kappa\mu}\, d\mu}{\sqrt{(1-2\mu x + x^2)}}$$
$$\equiv \frac{\kappa}{2\sinh\kappa}\int_{-1}^{1} e^{\frac{1}{2}\kappa x(1-\tau^2)+\kappa\tau}\, d\tau,$$
by the substitution $1-2\mu x + x^2 = (1-x\tau)^2$. Hence
$$\lambda_n = \frac{\kappa}{2\sinh\kappa}\int_{-1}^{1} \frac{[\frac{1}{2}\kappa(1-\tau^2)]^n}{n!} e^{\kappa\tau}\, d\tau$$
$$\sim \frac{1}{2}\int_{-1}^{1} \frac{(1-\tau^2)^n\, d\tau}{n!}\left(\frac{\kappa}{2}\right)^n \quad \text{as} \quad \kappa \to 0.$$

is least. The behaviour of this maximum difference as a function of V is shown in figure 1 and we see that, even in the worst case, it barely exceeds 0·02. We also find that the comparison (25) is surprisingly accurate. This is illustrated in figure 2 where the full line gives

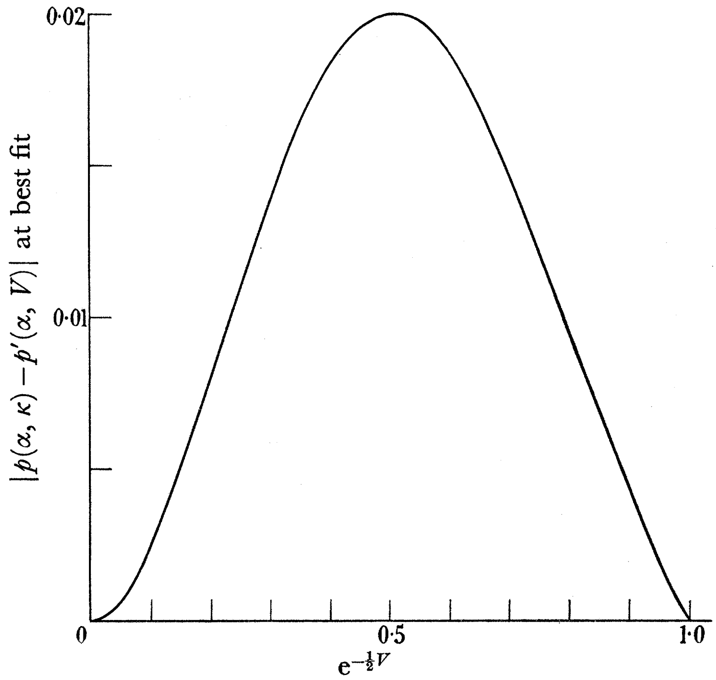

FIGURE 1. The maximum difference at best fit between the cumulative probability functions of Fisher's distribution and the Brownian distribution.

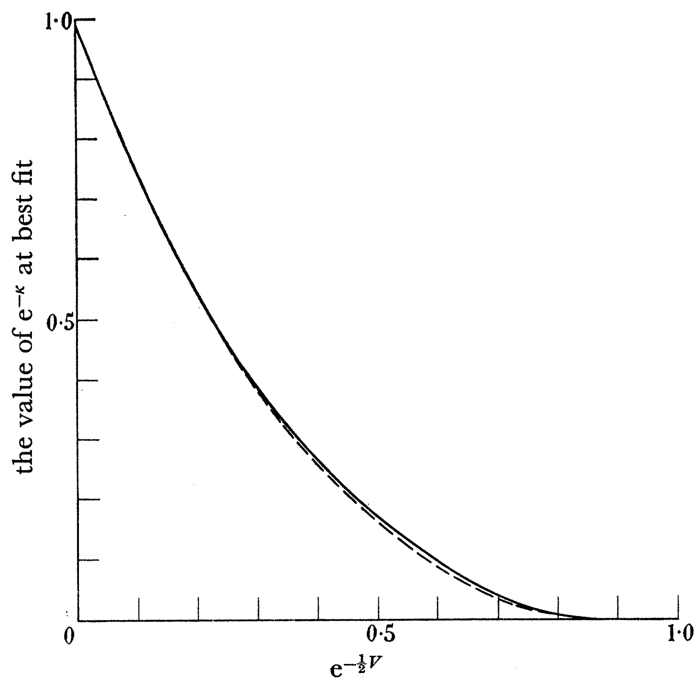

FIGURE 2. The relationship between κ and V for which the cumulative probability functions of Fisher's distribution and the Brownian distribution fit best: ——, from numerical calculations; – – – – –, from equation (25).

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the value of κ at best fit, plotted as a function of V, while the dashed line gives the value of κ implied by equation (25). However, the *percentage* error made in accepting estimate (25) exceeds 10 % for $e^{-\frac{1}{2}V}$ between 0·8 and 0·9. The function $p'(\alpha, V)$ is shown in table 1.

Table 1. The Brownian function for a sphere

$\cos\alpha$	$e^{-\frac{1}{2}V}=0·1$	$e^{-\frac{1}{2}V}=0·2$	$e^{-\frac{1}{2}V}=0·3$	$e^{-\frac{1}{2}V}=0·4$	$e^{-\frac{1}{2}V}=0·5$	$e^{-\frac{1}{2}V}=0·6$	$\cos\alpha$	$e^{-\frac{1}{2}V}=0·7$	$e^{-\frac{1}{2}V}=0·8$	$\cos\alpha$	$e^{-\frac{1}{2}V}=0·9$
0·9	0·064464	0·080226	0·098708	0·121746	0·152188	0·195161	0·95	0·139390	0·209104	0·975	0·215345
0·8	0·127360	0·156902	0·190974	0·232484	0·285651	0·357412	0·90	0·261310	0·377044	0·950	0·385589
0·7	0·188697	0·230091	0·277035	0·332894	0·402166	0·491434	0·85	0·367708	0·511417	0·925	0·519908
0·6	0·248480	0·299854	0·357123	0·423626	0·503390	0·601362	0·80	0·460341	0·618514	0·900	0·625674
0·5	0·306719	0·366255	0·431464	0·505308	0·590859	0·690838	0·75	0·540791	0·703523	0·875	0·708780
0·4	0·363420	0·429355	0·500284	0·578541	0·665998	0·763050	0·70	0·610481	0·770711	0·850	0·773944
0·3	0·418591	0·489216	0·563802	0·643905	0·730129	0·820786	0·60	0·722555	0·864982	0·825	0·824929
0·2	0·472240	0·545898	0·622233	0·701953	0·784468	0·866467	0·50	0·805253	0·922270	0·800	0·864733
0·1	0·524373	0·599464	0·675790	0·753218	0·830139	0·902188	0·40	0·865506	0·956360	0·775	0·895738
0·0	0·575000	0·649972	0·724681	0·798208	0·868175	0·929747	0·30	0·908790	0·976177	0·750	0·919833
−0·1	0·624126	0·697484	0·769110	0·837412	0·899520	0·950693	0·20	0·939400	0·987399	0·725	0·938515
−0·2	0·671760	0·742059	0·809277	0·871295	0·925042	0·966327	0·10	0·960666	0·993569	0·700	0·952964
−0·3	0·717909	0·783756	0·845379	0·900301	0·945527	0·977755	0·00	0·975147	0·996850	0·650	0·972691
−0·4	0·762580	0·822635	0·877609	0·924856	0·961692	0·985901	−0·10	0·984782	0·998530	0·600	0·984319
−0·5	0·805781	0·858755	0·906155	0·945363	0·974186	0·991529	−0·20	0·991022	0·999353	0·500	0·995009
−0·6	0·847520	0·892174	0·931204	0·962208	0·983593	0·995267	−0·30	0·994937	0·999734	0·400	0·998357
−0·7	0·887804	0·922951	0·952937	0·975756	0·990435	0·997621	−0·40	0·997300	0·999899	0·300	0·999569
−0·8	0·926640	0·951142	0·971531	0·986354	0·995182	0·998996	−0·60	0·999396	0·999990	0·200	0·999885
−0·9	0·964037	0·976806	0·987162	0·994332	0·998249	0·999708	−0·80	0·999926	1·000000	0·000	0·999993

From the statistical standpoint, Fisher's distribution is of great practical utility. The pole of maximum likelihood is independent of κ and its position together with R, the length of the resultant vector defined by the sample, provide exhaustive estimation. It does not possess the multiplicative property so characteristic of the plane Gaussian distribution and, from the abstract point of view, the Brownian distribution, which does, is a true analogue of the Gaussian distribution. However, exhaustive estimation does not seem to be possible for the Brownian distribution and, in fact, even the problem of determining the most likely pole and value of V is one of unprofitable difficulty. It is therefore a matter of some satisfaction that the agreement between the two distributions is so good. It is known that conclusions drawn on the hypothesis of a Gaussian distribution are insensitive to small modifications of it. We may presume, in a like manner, that results deduced on the assumption of Fisher's distribution would be affected little by the modification necessary to bring it to Brownian form and that, in practice, we may use Fisher's distribution even in those rare cases in which we are certain that the dispersion is dominated by a physical mechanism producing small random errors.

3. Convergence

We have in all cases

$$\lambda_0 = 1. \tag{26}$$

For $n > 0$,

$$|P_n(\cos\alpha)| < 1 \quad \text{if} \quad 0 < \alpha < \pi,$$
$$|\lambda_n| \leqslant 1, \tag{27}$$

with *strict* inequality in (27) except in certain extreme cases. This fact suggests that, although (9) may be only a formal expansion, the series (10) or (11) may be strictly convergent after a few random steps, as (14) is for any $V > 0$.

Now

$$S_n(\mathbf{r}') = \frac{2n+1}{4\pi} \iint \rho_0(\mathbf{r})\, P_n(\cos\theta)\, d\omega, \tag{28}$$

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where θ is the angle $\widehat{\mathbf{rr}'}$, or $\cos\theta = \mathbf{r}.\mathbf{r}'$ if we take the radius of the sphere to be unity and $\mathbf{r}$, $\mathbf{r}'$ to be the unit vectors drawn from the centre of the sphere to the points on the surface. From (28),

$$|S_n(\mathbf{r}')| \leqslant (2n+1)\,Q_0/4\pi = O(n). \tag{29}$$

This bound holds also for singular distributions with infinite density, but subject to the analogue of (2). No better result holds with this generality, as (17) shows: but better results can be obtained if ρ_0 is bounded. For

$$P_n(\cos\alpha) = \frac{1}{2\pi \mathrm{i}}\int_C \frac{\mathrm{d}z}{z^{n+1}\sqrt{(1-2z\cos\alpha+z^2)}},$$

the path C of integration separating the origin from the singularities at $z = \pm \mathrm{e}^{\mathrm{i}\alpha}$. C may be deformed into both sides of each of two radial cuts from $\pm\mathrm{e}^{\mathrm{i}\alpha}$ to infinity. This gives

$$\begin{aligned}|P_n(\cos\alpha)| &\leqslant \frac{2}{\pi}\int_1^\infty \frac{\mathrm{d}r}{r^{n+1}\sqrt{[2\sin\alpha(r-1)]}}\\ &= \frac{2}{\pi}\sqrt{\left(\frac{2}{\sin\alpha}\right)}\int_0^\infty \frac{\mathrm{d}t}{(1+t^2)^{n+1}}\\ &= \frac{2}{\pi}\sqrt{\left(\frac{2}{\sin\alpha}\right)}\int_0^{\frac{1}{2}\pi} \cos^{2n}\theta\,\mathrm{d}\theta\\ &< \frac{2}{\pi}\sqrt{\left(\frac{2}{\sin\alpha}\right)}\int_0^{\frac{1}{2}\pi} \mathrm{e}^{-n\theta^2}\mathrm{d}\theta\\ &< \sqrt{\left(\frac{2}{n\pi\sin\alpha}\right)}.\end{aligned} \tag{30}$$

And (30) easily leads to $S_n = O(\sqrt{n})$ if ρ_0 is bounded.

(*a*) For our basic case, in which each step is of the same fixed length α $(0<\alpha<\pi)$, (30) shows that

$$\lambda_n = O(n^{-\frac{1}{2}}) \tag{31}$$

and hence the series for ρ_t is absolutely and uniformly convergent for $t \geqslant 5$.

Next, suppose that the third extension is applied: and that the distribution function $p(\alpha)$ satisfies a Lipschitz condition at each end of the range,

$$p(\alpha) < K\alpha^\delta, \quad 1-p(\pi-\alpha) < K\alpha^\delta, \tag{32}$$

with some positive value of δ. This is quite a mild condition on $p(\alpha)$, although of course it is stronger than continuity at $\alpha = 0$ and $\alpha = \pi$. In (8) we have, by (30),

$$|P_n(\cos\alpha)| \leqslant \min\,(1, C/\sqrt{(\alpha n)}),$$

C denoting a constant not necessarily the same in all cases. Hence

$$\left.\begin{aligned}\lambda_n &= O(n^{-\frac{1}{2}}) &&\text{if}\quad \delta > \tfrac{1}{2},\\ \lambda_n &= O(n^{-\frac{1}{2}}\log n) &&\text{if}\quad \delta = \tfrac{1}{2},\\ \lambda_n &= O(n^{-\delta}) &&\text{if}\quad \delta < \tfrac{1}{2}.\end{aligned}\right\} \tag{33}$$

Thus if $\delta \geqslant \frac{1}{2}$ the series for ρ_t is again absolutely and uniformly convergent for $t \geqslant 5$. For $\delta < \frac{1}{2}$, absolute and uniform convergence may be delayed until $t\delta > 2$, but will be achieved after a finite number of steps.

If the second extension is applied, the steps satisfying (32) but with different values

$$\delta_1, \delta_2, \ldots, \delta_t, \ldots$$

of the order δ of the Lipschitz condition, absolute and uniform convergence of the series for ρ_t is attained as soon as

$$\delta_1' + \delta_2' + \ldots + \delta_t' > 2, \tag{34}$$

where δ_t' is the smaller of δ_t and $\frac{1}{2}$. In applying this rule, we can ignore any step which does not satisfy (32) with a positive value of δ; in other words we interpret δ, δ' for such a step as zero. This follows at once from (27).

The above, which assumes more than continuity of $p(\alpha)$ at 0 and π, does not involve any assumption of continuity between 0 and π.

(*b*) We now relax the conditions on $p(\alpha)$, assuming only that

$$p(\alpha) \text{ is continuous at } \alpha = 0 \text{ and } \alpha = \pi. \tag{35}$$

In this case we obtain

$$\lambda_n \to 0 \quad \text{as} \quad n \to \infty. \tag{36}$$

The factors λ_n *may* therefore change a non-convergent series into a convergent one. But (35) is not enough for *general* results of this nature.

To show this we construct in appendix A a function $p(\alpha)$, continuous in the whole range from 0 to π, including the endpoints, and increasing monotonically in that range from 0 to 1, for which

$$\lambda_n > (\log n)^{-1} \tag{37·1}$$

(not for all n but) for an infinity of values of n,

$$n = n_1, n_2, \ldots. \tag{37·2}$$

If we apply t steps of the random walk corresponding to $p(\alpha)$ to the initial distribution (17) we get

$$\rho_t = \Sigma \frac{2n+1}{4\pi} \lambda_n^t P_n(\cos\alpha),$$

and at $\theta = 0$ the terms of the series are not even bounded, *a fortiori* the series is not convergent, and that however great t may be. The like effect can be obtained with an initial distribution given by a bounded ρ_0.

(*c*) Finally, let us remove the assumption (35) and suppose instead that the random step assigns probabilities β, β' to $\alpha = 0$, $\alpha = \pi$ respectively. Then in place of (36) we get

$$\lambda_{2n} \to \beta + \beta', \quad \lambda_{2n+1} \to \beta - \beta' \quad \text{as} \quad n \to \infty. \tag{38}$$

Of these two limits, $\beta - \beta'$ *may* be zero, but our present hypothesis is that $\beta + \beta'$ is *not* zero, otherwise we should be in case (*b*).

Consider the special case in which $p(\alpha) = \beta$ for $0 < \alpha < \pi$, i.e. zero probability is assigned to all intermediate values of α. Here

$$\lambda_{2n} = \beta + \beta' = 1, \quad \lambda_{2n+1} = \beta - \beta' = 2\beta - 1. \tag{39}$$

Hence

$$\rho_1 = \Sigma S_{2n} + (2\beta - 1)\, \Sigma S_{2n+1}. \tag{40}$$

The result of t steps of this type is

$$\rho_t = \Sigma S_{2n} + A_t \Sigma S_{2n+1}, \tag{41}$$

where A_t is the product of the t factors $(2\beta-1)$, possibly varying from step to step; and the 'sum' of (41) is

$$\rho_t(\mathbf{r}) = \tfrac{1}{2}(1+A_t)\,\rho_0(\mathbf{r})+\tfrac{1}{2}(1-A_t)\,\rho_0(-\mathbf{r}). \tag{42}$$

From (41) it is clear that if there is a step for which $\beta=\beta'=\frac{1}{2}$ then the odd harmonics are completely eliminated at that step, and no further change takes place. If there is no step giving $\beta=\beta'$, the series for $\rho_t(\mathbf{r})$ converges for two distinct values of t if and only if the series for $\rho_0(\mathbf{r})$ and $\rho_0(-\mathbf{r})$ both converge, and the series for $\rho_t(\mathbf{r})$ converges for all $\mathbf{r}$ (and a fixed t) if and only if the series for $\rho_0(\mathbf{r})$ converges for all $\mathbf{r}$.

We have not established such precise results as these in the general case to which (38) refers. There may be examples $(p(\alpha),\rho_0)$ with $p(\alpha)$ discontinuous at the ends of the range and in which the harmonic expansion of ρ_1 is convergent (or is uniformly convergent) although that of ρ_0 is not. But let us say that a series has dominated convergence if it satisfies the M-test of Weierstrass,

$$|S_n(\mathbf{r})| \leqslant M_n, \quad \Sigma M_n < \infty. \tag{43}$$

Then in (a) a finite number of steps produced dominated convergence, whatever the initial distribution: in (b) this would occur with *some* distributions ρ_0: while in our present cases dominated convergence of the series of even harmonics cannot be produced (i.e. will not occur for ρ_t unless already occurring for ρ_0), and dominated convergence of the series of odd harmonics can only be produced if there is a step for which $\beta=\beta'$.

It is to be noted that if, for some k, ρ_k has a harmonic expansion which is uniformly convergent then, for any $t\geqslant k$, the harmonic expansion of ρ_t is also uniformly convergent, and with the same moduli of convergence as for ρ_k. This follows from the fact that each random step is an averaging process. For if

$$\rho_{t+1}(\mathbf{r}) = \frac{1}{2\pi}\iint \rho_t(\mathbf{r}')\,\mathrm{d}\psi\,\mathrm{d}p_{t+1}(\alpha),$$

(α,ψ) being spherical polar co-ordinates of $\mathbf{r}'$ with $\mathbf{r}$ as pole, then a like relation holds between corresponding terms

$$\Lambda_{n,t+1}S_n(\mathbf{r}) \quad \text{and} \quad \Lambda_{n,t}S_n(\mathbf{r}')$$

in the harmonic expansions of $\rho_{t+1}(\mathbf{r})$ and $\rho_t(\mathbf{r}')$. If then

$$\left|\sum_{\nu=n+1}^{\nu=n+q}\Lambda_{\nu,t}S_\nu(\mathbf{r}')\right| \leqslant \epsilon_n \quad \text{for all } \mathbf{r}' \text{ and all } q>0, \tag{44}$$

it follows that also

$$\left|\sum_{\nu=n+1}^{\nu=n+q}\Lambda_{\nu,t+1}S_\nu(\mathbf{r})\right| \leqslant \epsilon_n \quad \text{for all } \mathbf{r} \text{ and all } q>0.$$

Thus if (44) holds for $t=k$ it holds for all $t\geqslant k$.

4. Asymptotic behaviour

In our basic case the harmonic expansion of $\rho_t(\mathbf{r})$ converges uniformly in $(\mathbf{r},t)$ for $t\geqslant 5$. Moreover the individual terms, other than the first, decay away to zero as $t\to\infty$. It follows that

$$\rho_t(\mathbf{r}) \to S_0 \quad \text{uniformly as} \quad t\to\infty. \tag{45}$$

Our concern now is to generalize this result as far as possible.

We observe first that there are cases in which

$$\rho_t(\mathbf{r}) \to S_0 \quad \text{formally as} \quad t \to \infty \tag{46}$$

(by which we mean that each of the remaining harmonics decays away) although the harmonic series for ρ_t never converges and ρ_t is never a distribution of bounded density. Indeed, if

$$\Lambda_{nt} \to 0 \quad \text{as} \quad t \to \infty \quad \text{for every } n > 0, \tag{47}$$

then (46) holds for every initial distribution ρ_0. Now if, for all t,

$$p_t(\alpha) = \tfrac{1}{2} \quad \text{for} \quad 0 < \alpha < \tfrac{1}{2}\pi,$$
$$p_t(\alpha) = 1 \quad \text{for} \quad \alpha \geqslant \tfrac{1}{2}\pi,$$

then (47) holds and hence also (46); but if ρ_0 is the singular distribution (17) then ρ_t is, for every t, a singular distribution, assigning a charge 2^{-t} to one isolated point. Thus we must expect to have to replace the uniform convergence of (45) by something more general, even when (46) holds.

We shall in fact show that if

$$\sum_t \int_0^\pi \sin^2\alpha \, \mathrm{d}p_t(\alpha) \quad \text{diverges} \tag{48}$$

then

$$\rho_t(\mathbf{r}) \to S_0 \quad \text{strongly as} \quad t \to \infty, \tag{49}$$

i.e.

$$\iint |\rho_t - S_0| \, \mathrm{d}\omega \to 0 \quad \text{as} \quad t \to \infty. \tag{49 bis}$$

Secondly, we observe that if there is insufficient motion, if the steps of the random walk give too much concentration on values of α near zero, then the remaining harmonics will not be eliminated. In such a case we cannot expect stronger results than

$$\left.\begin{aligned} \Lambda_{nt} \to \Lambda_n \quad &\text{as} \quad t \to \infty, \\ \rho(\mathbf{r}) &= \Sigma \Lambda_n S_n, \\ \rho_t(\mathbf{r}) \to \rho(\mathbf{r}) \quad &\text{strongly as} \quad t \to \infty. \end{aligned}\right\} \tag{50}$$

Thirdly, we observe that a step in which α is certainly equal to π would turn $\rho_t(\mathbf{r})$ into $\rho_t(-\mathbf{r})$, while a step in which there is a high probability that α is near to π will turn $\rho_t(\mathbf{r})$ into a distribution which is in some sense near to $\rho_t(-\mathbf{r})$. Thus (50) cannot hold in all cases. What does hold in all cases is that there is a sequence of signs θ_t, and a limit distribution $\rho(\mathbf{r})$, such that

$$\left.\begin{aligned} \theta_t = \pm 1, \quad \Theta_t &= \theta_1 \theta_2 \ldots \theta_t = \pm 1, \\ \rho(\mathbf{r}) &= \Sigma \Lambda_n S_n, \\ \Theta_t^n \Lambda_{nt} &\to \Lambda_n, \\ \rho_t(\Theta_t \mathbf{r}) \to \rho(\mathbf{r}) \quad &\text{strongly.} \end{aligned}\right\} \tag{51}$$

An alternative expression of the result is as follows. We divide ρ_t into its even part σ_t and its odd part σ'_t, thus:

$$\left.\begin{aligned} \sigma_t(\mathbf{r}) &= \tfrac{1}{2}[\rho_t(\mathbf{r}) + \rho_t(-\mathbf{r})] = \Sigma \Lambda_{2n,t} S_{2n}, \\ \sigma'_t(\mathbf{r}) &= \tfrac{1}{2}[\rho_t(\mathbf{r}) - \rho_t(-\mathbf{r})] = \Sigma \Lambda_{2n+1,t} S_{2n+1}. \end{aligned}\right\} \tag{52}$$

We divide ρ similarly into its even and odd parts σ and σ'. The same random walk process which develops $\rho_1, \rho_2, \ldots$ from ρ_0 will develop $\sigma_1, \sigma_2, \ldots$ from σ_0 and will develop $\sigma'_1, \sigma'_2, \ldots$ from σ'_0. For the even parts we have simply

$$\left.\begin{aligned} \Lambda_{2n,t} &\to \Lambda_{2n}, \\ \sigma_t(\mathbf{r}) &\to \sigma(\mathbf{r}) \quad \text{strongly.} \end{aligned}\right\} \tag{53}$$

For the odd parts there is a sequence of signs Θ_t such that

$$\left.\begin{aligned} \Theta_t \Lambda_{2n+1,t} &\to \Lambda_{2n+1}, \\ \sigma'_t(\Theta_t \mathbf{r}) = \Theta_t \sigma'_t(\mathbf{r}) &\to \sigma'(\mathbf{r}) \quad \text{strongly.} \end{aligned}\right\} \tag{54}$$

Conditions for the simpler results (50) will appear in the course of the proofs below. We remark at once that from the above it will follow that among the distributions ρ_t

formal convergence implies strong convergence. (55)

4·1. We begin with a theorem which has its own interest and will be a principal tool in what follows.

THEOREM. *Suppose that, for some particular value of t, ρ_t is a continuous density. Then the later distributions $\rho_{t+\tau}$ are all continuous densities with the same moduli of continuity as ρ_t, they are uniformly bounded, and among them formal convergence implies uniform convergence.*

Suppose that

$$|\rho_t(\mathbf{r}') - \rho_t(\mathbf{r}'_1)| \leqslant \epsilon$$

whenever the angular distance $\widehat{\mathbf{r}'\mathbf{r}'_1}$ is $\leqslant \delta$. Let $\mathbf{r}$, $\mathbf{r}_1$ be any two points on the sphere with angular distance $\widehat{\mathbf{r}\mathbf{r}_1} = \alpha \leqslant \delta$. Then there is a rotation U of angle α which turns the sphere into itself and turns the point $\mathbf{r}$ into the point $\mathbf{r}_1$. Then $\rho_{t+1}(\mathbf{r}_1) = \rho_{t+1}(U\mathbf{r})$ is the same average of the values $\rho_t(U\mathbf{r}')$ as $\rho_{t+1}(\mathbf{r})$ is of the values $\rho_t(\mathbf{r}')$. Since the angular separation of corresponding points $\mathbf{r}'$, $U\mathbf{r}$ is $\leqslant \alpha \leqslant \delta$, it follows that

$$|\rho_{t+1}(\mathbf{r}) - \rho_{t+1}(\mathbf{r}_1)| \leqslant \epsilon.$$

Thus ρ_{t+1} possesses the same moduli of continuity as ρ_t. In particular, the distribution ρ_{t+1} is one of continuous density. Repeating the argument, we see that the densities $\rho_{t+\tau}$ are equicontinuous.

Since they all have the same mean S_0, they are also uniformly bounded.

Since they are uniformly bounded, we can by the diagonal selection process find a subsequence $\{\rho_{t_\nu}\}$ which is convergent at each of a given sequence of points $\{\mathbf{r}_k\}$ on the sphere.

We may choose a sequence $\{\mathbf{r}_k\}$ everywhere dense on the sphere. Since the functions $\rho_{t_\nu}(\mathbf{r})$ are equicontinuous, it then follows that they converge at every point $\mathbf{r}$ on the sphere, uniformly in $\mathbf{r}$. Let the limit function be $\rho(\mathbf{r})$, then $\rho(\mathbf{r})$ also possesses the same moduli of continuity as the original $\rho_t(\mathbf{r})$, and the functions $\rho_{t_\nu}(\mathbf{r})$ converge formally as well as uniformly to $\rho(\mathbf{r})$.

Now suppose for the moment that it is *not* true that the complete sequence $\{\rho_{t+\tau}(\mathbf{r})\}$ converges uniformly to $\rho(\mathbf{r})$. Then there is a positive constant c and a subsequence $\{\rho_{t'_\nu}(\mathbf{r})\}$ such that, for each ν,

$$\max_{(\mathbf{r})} |\rho_{t'_\nu}(\mathbf{r}) - \rho(\mathbf{r})| \geqslant c.$$

By further selection we may secure that also the sequence $\{\rho_{t_i'}\}$ converges uniformly to some limit $\rho'(\mathbf{r})$. Clearly $\rho'(\mathbf{r})$ is not the same function as $\rho(\mathbf{r})$. Since both are continuous functions they do not have the same harmonic series. Hence it is *not* true that the complete sequence $\{\rho_{t+\tau}\}$ converges formally.

We have shown that if the complete sequence $\{\rho_{t+\tau}\}$ does not converge uniformly then it does not converge formally. In other words, if it does converge formally then it must also converge uniformly. By the same argument, any subsequence of the $\rho_{t+\tau}$ which converges formally must also converge uniformly: thus among the functions $\rho_{t+\tau}$ formal convergence implies uniform convergence.

4·2. For each $n > 0$ there is a positive constant k_n such that

$$1-|P_n(\cos\alpha)| \geqslant k_n \sin^2\alpha,$$

$$1-|\lambda_{nt}| \geqslant k_n \int_0^\pi \sin^2\alpha \, dp_t(\alpha).$$

We suppose now that (48) holds.

Then the product $\prod_{(t)} \lambda_{nt}$ diverges (to zero) for each $n > 0$, so that (47) and (46) follow. We note also that

$$\prod_{\tau=1}^{T} \lambda_{n,\,t+\tau} \to 0 \quad \text{as} \quad T\to\infty \tag{56}$$

for every t and every $n > 0$. This simply says that the property (47), which formally eliminates all harmonics other than S_0, is still possessed by the random walk even when the first t steps are removed. Thus (47) does not depend on any 'accidental' vanishing of an individual factor λ_{nt}.

We denote by β_t, β_t' the probabilities assigned to $\alpha = 0$, $\alpha = \pi$ in the tth step, i.e.

$$\beta_t = p_t(0+0), \quad \beta_t' = 1-p_t(\pi-0).$$

Either or both of these quantities may of course be zero for any particular value of t. From (48) it follows that

$$\sum_{(t)} (1-\beta_t-\beta_t') = \sum_{(t)} \int_{0+0}^{\pi-0} dp_t(\alpha) \geqslant \sum_{(t)} \int_0^\pi \sin^2\alpha \, dp_t(\alpha) = \infty,$$

so that $\prod_{(t)} (\beta_t+\beta_t')$ diverges to zero, and indeed

$$\prod_{\tau=1}^{T} (\beta_{t+\tau}+\beta'_{t+\tau}) \to 0 \quad \text{as} \quad T\to\infty \tag{57}$$

for every t.

Given $\epsilon > 0$ we can find an integer k and a positive number δ such that

$$\int \dots \int_V dp_1(\alpha_1) \dots dp_k(\alpha_k) < \epsilon, \tag{58}$$

where V is the region in which at most four of the angles $\alpha_1, \dots, \alpha_k$ have sines not less than δ.

To see this we choose $\eta > 0$ so small that $(1-\eta)^5 > 1-\epsilon$. We then choose t_1 so large that

$$\prod_{t=1}^{t_1} (\beta_t+\beta_t') < \eta,$$

then $\delta_1 > 0$ so small that the chance that $\sin\alpha_t < \delta_1$ for all $t \leqslant t_1$ is also less than η. Thus we have in the space of co-ordinates $(\alpha_1, \dots, \alpha_{t_1})$ a region U_1, of probability at least $1-\eta$, at every point of which there is at least one t $(1 \leqslant t \leqslant t_1)$ for which $\sin\alpha_t \geqslant \delta_1$.

Similarly we choose t_2, $\delta_2 > 0$ so that in the space of co-ordinates $(\alpha_t;\, t_1 < t \leqslant t_2)$ there is a region U_2, of probability at least $(1-\eta)$, at every point of which there is at least one t $(t_1 < t \leqslant t_2)$ for which $\sin\alpha_t \geqslant \delta_2$.

We make three more constructions on these lines, put $k = t_5$, define δ to be the least of $\delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ and U to be the region complementary to V. Then U, the region in which $\sin\alpha_t \geqslant \delta$ for at least five values of t in the range, certainly includes the product

$$U_1 \times U_2 \times U_3 \times U_4 \times U_5,$$

and so is of probability $$p_U > (1-\eta)^5 > 1-\epsilon,$$

whence (58) follows.

Let $\rho_{k;\,\alpha_1,\ldots}$ be the result of steps random in direction but of fixed lengths $\alpha_1, \alpha_2, \ldots, \alpha_k$ on the original distribution ρ_0 satisfying (2). If $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is in U then $\rho_{k;\,\alpha_1,\ldots}$ has a harmonic series which is convergent uniformly in both $\mathbf{r}$ and the α_t, the harmonic of order n being in modulus

$$\leqslant \frac{2n+1}{4\pi} Q_0 \left(\frac{2}{n\pi\delta}\right)^{\frac{5}{2}}.$$

It follows that $$\rho_{k;\,U} \equiv \underset{(U)}{\int\ldots\int} \rho_{k;\,\alpha_1,\ldots,\,\alpha_k}\, \mathrm{d}p_1(\alpha_1) \ldots \mathrm{d}p_k(\alpha_k) \tag{59}$$

is continuous. Now $$\rho_k = \underset{(U+V)}{\int\ldots\int} \rho_{k;\,\alpha_1,\ldots,\,\alpha_k}\, \mathrm{d}p_1(\alpha_1) \ldots \mathrm{d}p_k(\alpha_k)$$

$$= \rho_{k;\,U} + \rho_{k;\,V}, \tag{60}$$

say, and $$\iint |\rho_{k;\,V}|\, \mathrm{d}\omega \leqslant p_V Q_0 = (1-p_U)\, Q_0 \leqslant \epsilon Q_0,$$

where p_V is the left-hand member of (58).

With an obvious notation we now have

$$\rho_t = \rho_{t;\,U} + \rho_{t;\,V} \quad \text{for all} \quad t > k.$$

By the theorem of § 4·1, $$\rho_{t:\,U} \to p_U S_0 \quad \text{uniformly as} \quad t \to \infty,$$

while $$\iint |\rho_{t;\,V}|\, \mathrm{d}\omega \leqslant \iint |\rho_{k;\,V}|\, \mathrm{d}\omega \leqslant \epsilon Q_0.$$

(49) now follows by familiar arguments.

We remark in conclusion that the factor $\sin^2\alpha$ in (48) may be replaced by $1-|\cos\alpha|$ or by $\alpha^2(\pi-\alpha)^2$.

4·3. For each n there is a positive constant k_n such that

$$1 - P_{2n+1}(\cos\alpha) \geqslant k_n(1-\cos\alpha),$$
$$1 + P_{2n+1}(\cos\alpha) \geqslant k_n(1+\cos\alpha).$$

Hence $$1-\lambda_{2n+1,\,t} \geqslant k_n \int_0^\pi (1-\cos\alpha)\, \mathrm{d}p_t(\alpha),$$

$$1+\lambda_{2n+1,\,t} \geqslant k_n \int_0^\pi (1+\cos\alpha)\, \mathrm{d}p_t(\alpha),$$

and so $$1-|\lambda_{2n+1,\,t}| \geqslant k_n \min_{\phi=\pm 1} \int_0^\pi (1-\phi\cos\alpha)\, \mathrm{d}p_t(\alpha). \tag{61}$$

We now no longer assume (48), but suppose instead that

$$\sum_{(t)} \min_{\phi=\pm 1} \int_0^\pi (1-\phi\cos\alpha)\,\mathrm{d}p_t(\alpha) \quad \text{diverges.} \tag{62}$$

Then

$$\prod_{(t)} \lambda_{2n+1,\,t} \quad \text{diverges to zero} \tag{63}$$

for every n. The same comment is to be made on (63) as has been made twice before in similar circumstances, namely, that the product still diverges to zero if a finite number of initial factors be omitted.

In this case the odd part σ_t' of ρ_t converges formally to zero, and we shall prove (50) with the additional fact that the limit distribution $\rho(\mathbf{r})$ is even:

$$\rho(\mathbf{r}) = \rho(-\mathbf{r}), \quad \Lambda_{2n+1} = 0. \tag{64}$$

Our proof, however, must distinguish two cases, according as (57) is or is not satisfied.

If (57) holds, we follow the argument of § 4·2 as far as (60). Now for every $n > 0$ there is a positive constant K_n such that

$$1 - P_{2n}(\cos\alpha) \leqslant K_n \sin^2\alpha,$$

$$1-\lambda_{2n,\,t} \leqslant K_n \int_0^\pi \sin^2\alpha\,\mathrm{d}p_t(\alpha). \tag{65}$$

We are entitled to assume that (48) does *not* hold, that

$$\sum_{(t)} \int_0^\pi \sin^2\alpha\,\mathrm{d}p_t(\alpha) \quad \text{converges.}$$

Accordingly (65) shows that the infinite product $\prod_{(t)} \lambda_{2n,\,t}$ is convergent (absolutely): this is still true if the first k factors be omitted.

From this and from (63) it follows that the distributions $\rho_{t;\,U}$ $(t \geqslant k)$ formed from the continuous-density distribution $\rho_{k;\,U}$ are formally convergent as $t \to \infty$. By the theorem of § 4·1, they are therefore uniformly convergent to some limit $\rho_U(\mathbf{r})$ as $t \to \infty$. Hence

$$|\rho_{t;\,U} - \rho_{t';\,U}| \leqslant \frac{\epsilon Q_0}{4\pi} \quad \text{for all } \mathbf{r},$$

$$\iint |\rho_{t;\,U} - \rho_{t';\,U}|\,\mathrm{d}\omega \leqslant \epsilon Q_0,$$

for all pairs t, t' sufficiently great,

$$t \geqslant T_0 = T_0(\epsilon), \quad t' \geqslant T_0. \tag{66}$$

But as in § 4·2, and with no restriction on t, t' except the obvious one that they be $\geqslant k$, we have

$$\iint |\rho_{t;\,V}|\,\mathrm{d}\omega \leqslant \epsilon Q_0, \quad \iint |\rho_{t';\,V}|\,\mathrm{d}\omega \leqslant \epsilon Q_0.$$

Hence

$$\iint |\rho_t - \rho_{t'}|\,\mathrm{d}\omega \leqslant 3\epsilon Q_0,$$

provided that t, t' satisfy (66). Hence the distributions ρ_t converge strongly to some limit distribution ρ. They therefore also converge formally to ρ, which completes the proof of (50) for this case: and (64) then follows from (63).

It is worth noting that
$$\iint |\rho_U - \rho| \, d\omega \leqslant \epsilon Q_0,$$
so that we can find a sequence of distributions of continuous density which also converge strongly to the distribution ρ. Thus the distribution ρ is one in which the charge is an *absolutely continuous* additive function of sets, and ρ itself a genuine density, albeit not in general bounded. By (63), ρ_U is an even distribution, and we see again that ρ is even.

4·4. We turn now to the case in which (62) holds, but *not* (57): thus
$$\sum_{(t)} (1-\beta_t-\beta_t') \quad \text{converges}, \tag{67·1}$$
$$\prod_{(t)} (\beta_t+\beta_t') \quad \text{converges}. \tag{67·2}$$
(67·1) states unambiguously what we mean: it is equivalent to (67·2) if, in interpreting (67·2), any finite number of (possibly vanishing) factors are ignored, and if, as is usual, a product tending to zero is counted as diverging. The terms in (67·1) being non-negative, the assumed convergence is in both cases absolute: and it excludes (48), since (48) implies (57).

Combining (67·1) with (62) we easily deduce that
$$\Sigma \min (\beta_t, \beta_t') \quad \text{diverges}$$
and hence that
$$\Pi |\beta_t - \beta_t'| \quad \text{diverges to zero}. \tag{68}$$
As before, (68) still holds if the first k factors in the product be omitted.

Now let $\epsilon > 0$ be given, and choose k so that
$$\prod_{t>k} (\beta_t+\beta_t') > 1-\epsilon.$$
We again separate ρ_t for $t \geqslant k$ into a main part $\rho_{t;\,U}$ and a small part $\rho_{t;\,V}$; thus
$$\rho_t = \rho_{t;\,U} + \rho_{t;\,V} \quad (t \geqslant k), \tag{69}$$
but the principle of separation is rather different. By $\rho_{t;\,U}$ we denote that part of ρ_t which is obtained from ρ_k if all the subsequent steps $\alpha_{k+1}, \ldots, \alpha_t$ are equal to 0 or π, thus
$$(t;\,U): \quad \alpha_\tau = 0 \quad \text{or} \quad \pi, \quad \text{for} \quad k < \tau \leqslant t.$$
Accordingly $\rho_{t;\,V}$ denotes that part of ρ_t which is obtained from ρ_k by steps $\alpha_{k+1}, \ldots, \alpha_t$ at least one of which differs both from 0 and from π. With an obvious notation
$$p_{t;\,U} = \prod_{k<\tau\leqslant t} (\beta_\tau+\beta_\tau') > 1-\epsilon,$$
$$p_{t;\,V} = 1-p_{t;\,U} < \epsilon.$$
We also separate ρ_k and $\rho_{t;\,U}$ into their even and odd parts: there is no advantage to be gained from a similar analysis of $\rho_{t;\,V}$. Then
$$\left.\begin{aligned} \rho_t &= \sigma_{t;\,U} + \sigma_{t;\,U}' + \rho_{t;\,V}, \\ \sigma_{t;\,U} &= p_{t;\,U}\, \sigma_k, \\ \sigma_{t;U}' &= \sigma_k' \prod_{k<\tau\leqslant t} (\beta_\tau - \beta_\tau'), \\ \iint |\rho_{t;\,V}| \, d\omega &\leqslant p_{t;\,V} \iint |\rho_k| \, d\omega. \end{aligned}\right\} \tag{70}$$

By (68), the product in (70) is numerically less than ϵ for $t > T_0 = T_0(\epsilon)$. If, then, both t and t' exceed T_0 we easily deduce

$$\iint |\rho_t - \rho_{t'}| \, d\omega \leqslant \epsilon \iint (|\sigma_k| + 2\,|\sigma'_k| + 2\,|\rho_k|) \, d\omega$$
$$\leqslant 5\epsilon Q_0.$$

It follows that the distribution ρ_t tends strongly to some limit distribution ρ, hence also formally. That ρ is an even distribution follows from (63): alternatively we easily see that

$$\iint |\rho - \sigma_k| \, d\omega \leqslant 2\epsilon Q_0,$$

so that ρ, as the strong limit of even distributions, is itself even.

4·5. We assume now that (62) is false,

$$\sum_{(t)} \min_{\phi=\pm 1} \int_0^\pi (1-\phi\cos\alpha)\, dp_t(\alpha) \quad \text{converges.} \tag{71}$$

Let the minimum for the tth term be given by $\phi = \theta_t$: there can be no ambiguity after the first few terms since the series converges. Then

$$\sum_{(t)} \int_0^\pi (1-\theta_t\cos\alpha)\, dp_t(\alpha) \quad \text{converges.} \tag{71 bis}$$

Define Θ_t as in (51), and introduce a new random walk† by

$$\rho'_t(\mathbf{r}) = \rho_t(\Theta\mathbf{r}),$$
$$p'_t(\alpha) = p_t(\alpha) \qquad \text{if} \quad \theta_t = +1,$$
$$= 1 - p_t(\pi-\alpha) \quad \text{if} \quad \theta_t = -1.$$

Then $\rho'_1, \rho'_2, \ldots$ are in fact formed from ρ'_0 $(= \rho_0)$ by the random steps given by the cumulative probability functions p'_t, and

$$\sum_{(t)} \int_0^\pi (1-\cos\alpha)\, dp'_t(\alpha) \quad \text{converges.} \tag{71 ter}$$

To prove (51) for ρ_t we have to prove (50) for ρ'_t, but there will be no question now of proving (64).

To avoid unnecessary complexities we now drop the dashes and suppose that

$$\sum_{(t)} \int_0^\pi (1-\cos\alpha)\, dp_t(\alpha) \quad \text{converges.} \tag{72}$$

For every $n > 0$ there is a positive constant K_n such that

$$1 - P_n(\cos\alpha) \leqslant K_n(1-\cos\alpha),$$

$$1 - \lambda_{nt} \leqslant K_n \int_0^\pi (1-\cos\alpha)\, dp_t(\alpha).$$

Hence, for every n, $\quad \prod_{(t)} \lambda_{nt}$ converges absolutely, (73)

† For $\theta_t = -1$, this might conflict with the convention adopted in the introduction which permits $p(\alpha)$ to be discontinuous to the left but not to the right. Evidently, however, this produces no essential difficulty.

which still applies if the first k factors be omitted and does not exclude the possibility that a finite number of factors (for each n) should vanish. Formal convergence

$$\Lambda_{nt} \to \Lambda_n \quad \text{as} \quad t \to \infty$$

is guaranteed by (73), and a Λ_n can vanish only 'accidentally' by the vanishing of an individual factor λ_{nt}.

We have again to treat separately the cases in which (57) holds and those in which (67) holds. We remark in passing that, with the adjustment leading to (72), $\Sigma\beta'_t$ converges, so that the separation is according to the divergence or convergence of

$$\Sigma(1-\beta_t), \quad \Pi\beta_t.$$

If (57) holds, we argue as in § 4·3. By (73), the main part of $\rho_{t;\,U}$ converges formally as $t \to \infty$, and so, by the theorem of § 4·1, it also converges uniformly, *a fortiori* strongly, to some limit ρ_U. The residues $\rho_{t;\,V}$ being of small total charge, it follows as in § 4·3 that the ρ_t converge strongly. Again as in § 4·3, the limit distribution can be obtained as the strong limit of distributions ρ_U of continuous density, so that ρ itself is a genuine density, albeit not in general bounded. The difference is that ρ_U, ρ need not be even.

If (67) holds, we argue broadly as in § 4·4: the argument is indeed simpler. $\Pi\beta_t$ converges, and we choose k so that

$$\prod_{t>k}\beta_t > 1-\epsilon,$$

where $\epsilon > 0$ is given. The basis of the separation

$$\rho_t = \rho_{t;\,U}+\rho_{t;\,V} \quad (t \geqslant k)$$

of ρ_t into the main and minor parts is expressed by

$$(t;\,U): \quad \alpha_\tau = 0 \quad \text{for} \quad k < \tau \leqslant t.$$

Then

$$\rho_{t;\,U} = p_{t;\,U}\rho_k,$$

$$p_{t;\,U} = \prod_{k<\tau\leqslant t}\beta_\tau > 1-\epsilon,$$

Then

$$\iint |\rho_{t;\,V}|\,\mathrm{d}\omega \leqslant p_{t;\,V}\iint |\rho_k|\,\mathrm{d}\omega \leqslant \epsilon Q_0,$$

$$\iint |\rho_t-\rho_k|\,\mathrm{d}\omega \leqslant 2\epsilon Q_0 \quad (\text{all } t \geqslant k)$$

and so ρ_t tends strongly to some limit distribution ρ. This completes the proof of (51) and so of (55): in particular, (47) implies (49).

4·6. We have shown that (48) implies (49), for every ρ_0, and hence (47). It is in a certain sense 'nearly true' that (47), conversely, implies (48). In fact (65) shows that if (48) does not hold then the product $\prod_{(t)}\lambda_{2n,\,t}$ converges absolutely, and so $\Lambda_{2n,\,t}$ tends to a non-zero limit as $t \to \infty$ unless, for some particular value of t, $\lambda_{2n,\,t}$ 'accidentally' vanishes. Thus we have the (apparently) rather strong result that if we know a single even value $2k$ for which $\Lambda_{2k,\,t} \to 0$, *other than 'accidentally'*, then we can deduce (48), (49), and also (47) for every n, whether odd or even.

But it is possible for (47) to hold 'accidentally' for *every* $n > 0$, even with a random walk $\{p_t(\alpha)\}$ satisfying (72). A trivial example of this is found if $p_1(\alpha) = \frac{1}{2}(1-\cos\alpha)$ since then, whatever the later $p_t(\alpha)$ may be, $\rho_t = S_0$ and $\Lambda_{nt} = 0$ for all $n > 0$ and $t > 0$. But there are non-trivial examples in which, for any finite value of t, only a finite number of the Λ_{nt} have been reduced to zero. In fact, given $\epsilon > 0$ we can construct a function $f(\mu)$, continuous and positive in $-1 \leqslant \mu \leqslant 1$, and such that

$$\int_{-1}^{1} f(\mu)\, \mathrm{d}\mu = 1, \quad \int_{-1}^{1} (1-\mu) f(\mu)\, \mathrm{d}\mu < \epsilon.$$

Since any continuous function can be approximated arbitrarily closely by a polynomial, we may suppose that $f(\mu)$ is a polynomial: and among the polynomials satisfying these conditions we choose one with the least possible degree $n_0(\epsilon)$.

Now let E be any selection, finite or infinite, of the integers $n > n_0(\epsilon)$, and define

$$\phi(\mu) = f(\mu) + \eta \sum_{n \in E} 2^{-n} P_n(\mu).$$

$\phi(\mu)$ is certainly continuous in $(-1, 1)$ and $\int_{-1}^{1} \phi(\mu)\, \mathrm{d}\mu = 1$. For sufficiently small η, $\phi(\mu) \geqslant 0$ in $(-1, 1)$ and

$$\int_{-1}^{1} (1-\mu)\, \phi(\mu)\, \mathrm{d}\mu < \epsilon.$$

Define $p(\alpha)$ by $\mathrm{d}p(\alpha) = \phi(\cos\alpha)\sin\alpha\, \mathrm{d}\alpha$, then $\lambda_n \neq 0$ if $n \in E$, $\lambda_n = 0$ if $n > n_0(\epsilon)$ and n does not belong to E. Thus we can satisfy the condition

$$\int_{0}^{\pi} (1-\cos\alpha)\, \mathrm{d}p(\alpha) < \epsilon$$

and still have complete control over which of the λ_n vanish, and which do not vanish, in $n > n_0(\epsilon)$.

This having been established, we take any convergent series $\Sigma\epsilon_t$ of decreasing positive terms, and we define $p_t(\alpha)$ as above with ϵ_t for ϵ and the set of all $n > n_0(\epsilon_{t+1})$ for E. Then $\Lambda_{nt} = 0$ if and only if $n \leqslant n_0(\epsilon_{t+1})$. Since $\epsilon_t \to 0$, therefore $n_0(\epsilon_t) \to \infty$, so that (47) holds, 'accidentally' for every n, and non-trivially. And (72) is satisfied since $\Sigma\epsilon_t$ converges.

In this example, ρ_t is a distribution of continuous density for all $t > 0$, whatever ρ_0 may be, and hence not only (49) but even (45) holds.

Accidents which happen always can scarcely be ignored: but the standing of our criteria (48), (62) is improved if we make the following observation. If (48) is satisfied, every harmonic but S_0 is eliminated *however many initial steps of the walk are omitted*: and there is no even harmonic of which this is true unless (48) is satisfied. If (62) is satisfied, every odd harmonic is eliminated *however many initial steps of the walk are omitted*: and there is no odd harmonic of which this is true unless (62) is satisfied. In this sense (62) is necessary as well as sufficient for the elimination of either one odd harmonic or all odd harmonics, while the stronger condition (48) is in this sense necessary as well as sufficient for the elimination of even harmonics.

5. Random walk on a closed Riemannian manifold

5·1. Random walk on a Riemannian manifold M can be expressed by a sequence of functions $\tau_t(\mathbf{r}, \mathbf{r}')$ such that, if the point has reached the position $\mathbf{r}'$ after $(t-1)$ steps, the probability that it is brought within the volume $\mathrm{d}V$ at $\mathbf{r}$ by the tth step is

$$\tau_t(\mathbf{r}, \mathbf{r}')\, \mathrm{d}V,$$

where
$$\mathrm{d}V = \mathrm{d}\mathbf{r}\sqrt{g(\mathbf{r})},$$

and $g(\mathbf{r})$ is the determinant of the metric tensor $g_{ij}(\mathbf{r})$ at $\mathbf{r}$. Clearly $\tau_t(\mathbf{r},\mathbf{r}')$ must satisfy

$$\int_M \tau_t(\mathbf{r},\mathbf{r}')\,\mathrm{d}V = 1. \tag{74}$$

In place of equation (4), we shall have

$$\rho_t(\mathbf{r}) = \int_M \tau_t(\mathbf{r},\mathbf{r}')\,\rho_{t-1}(\mathbf{r}')\,\mathrm{d}V', \tag{75}$$

and the characteristic functions ϕ_{nt} of the tth step therefore satisfy

$$\int_M \tau_t(\mathbf{r},\mathbf{r}')\,\phi_{nt}(\mathbf{r}')\,\mathrm{d}V' = \lambda_{nt}\phi_{nt}(\mathbf{r}). \tag{76}$$

We shall suppose that $\tau_t(\mathbf{r},\mathbf{r}')$ is symmetrical in $\mathbf{r}$, $\mathbf{r}'$ (cf. (84) et seq.). Note that, in this case, equation (75) is exactly analogous to (4) in that it expresses $\rho_t(\mathbf{r})$ as a (weighted) *average* of $\rho_{t-1}(\mathbf{r}')$. Also, equation (76) shows that $\phi_{nt}(\mathbf{r})$, $\phi_{mt}(\mathbf{r})$ are orthogonal† if their characteristic values are different:

$$\int_M \phi_{nt}(\mathbf{r})\,\phi_{mt}(\mathbf{r})\,\mathrm{d}V = 0. \tag{77}$$

In any case of degeneracy, a base of mutually orthogonal functions can be constructed for those characteristic functions belonging to the same characteristic value. Thus, we may consider the set to be orthogonal and, without loss of generality, normalized to unity:

$$\int_M [\phi_{nt}(\mathbf{r})]^2\,\mathrm{d}V = 1. \tag{78}$$

Provided that the set is complete, equation (76) shows that $\tau_t(\mathbf{r},\mathbf{r}')$ possesses the (formal) expansion

$$\tau_t(\mathbf{r},\mathbf{r}') = \sum_n \lambda_{nt}\phi_{nt}(\mathbf{r})\,\phi_{nt}(\mathbf{r}'), \tag{79}$$

and, if the probability density $\rho_{t-1}(\mathbf{r}')$ before the tth step is

$$\rho_{t-1}(\mathbf{r}') = \sum_n C_n\phi_{nt}(\mathbf{r}'), \tag{80}$$

then the probability density $\rho_t(\mathbf{r})$ after the tth step is

$$\rho_t(\mathbf{r}) = \sum_n \lambda_{nt}C_n\phi_{nt}(\mathbf{r}). \tag{81}$$

It has been observed in § 2 that the success of our method for random walk on a sphere rests on the fact that the same characteristic functions occur for each value of t. In the same

† If $\tau_t(\mathbf{r},\mathbf{r}')$ is not symmetric and if ψ_{mt} are characteristic functions of $\tau_t(\mathbf{r}',\mathbf{r})$, the functions ϕ_{nt}, ψ_{mt} would generally form a biorthonormal set:

$$\int_M \phi_{nt}(\mathbf{r})\,\psi_{mt}(\mathbf{r})\,\mathrm{d}V = \delta_{mn},$$

leading to a very similar analysis.

way in the present case, if ϕ_{nt} is independent† of t, the formal expansion proceeds in exact analogy, and the distribution after t steps from a starting point $\mathbf{r}_0$ is

$$\rho_t(\mathbf{r}) = \sum_n \Lambda_{nt}\phi_n(\mathbf{r}_0)\,\phi_n(\mathbf{r}), \tag{82}$$

where

$$\Lambda_{nt} = \prod_{k=1}^{t} \lambda_{nk}. \tag{12}$$

5·2. We have shown that from an (orthonormal) base, we can construct a class of probability functions, each of which is characterized by a diagonal form (79), and for which our basic method succeeds. However, such probability functions do not necessarily correspond to our notions of 'random walk', and these will now be clarified.

Along each geodesic through $\mathbf{r}'$ mark off a distance α, thereby obtaining a 'geodesic sphere', centre $\mathbf{r}'$ and radius α, whose surface and volume will be denoted by $S(\mathbf{r}',\alpha)$ and $V(\mathbf{r}',\alpha)$. A random walk of length α from $\mathbf{r}'$ is one which with certainty moves the point $\mathbf{r}'$ to a point on $S(\mathbf{r}',\alpha)$, the probability that it moves to any infinitesimal patch $\mathrm{d}S$ of $S(\mathbf{r}',\alpha)$ being $\mathrm{d}\omega/\Omega$, where $\mathrm{d}\omega$ is the infinitesimal solid angle subtended at $\mathbf{r}'$ by the geodesics through $\mathbf{r}'$ and $\mathrm{d}S$, and

$$\Omega = 2\pi^{\frac{1}{2}q}/\Gamma(\tfrac{1}{2}q)$$

is the total solid angle, M being of q dimensions. The general random walk (third extension) is obtained by letting α itself have a probability distribution $\mathrm{d}p(\alpha)$.

Let $\Phi(\mathbf{r},\mathbf{r}')$ be defined by

$$\mathrm{d}S = \Phi(\mathbf{r},\mathbf{r}')\,\mathrm{d}\omega. \tag{83}$$

Strictly $\Phi(\mathbf{r},\mathbf{r}')$ is a functional of the geodesic arc and not merely of its endpoints. It is, by convention, positive for small α, but it may change sign at a conjugate focus. The relationship between $\rho_{t-1}(\mathbf{r}')$ and $\rho_t(\mathbf{r})$ is

$$\rho_t(\mathbf{r}) = \frac{1}{\Omega}\int_{S(\mathbf{r},\alpha)} \frac{\rho_{t-1}(\mathbf{r}')\,\mathrm{d}S'}{\Phi(\mathbf{r},\mathbf{r}')}, \tag{84}$$

and, since $\Phi(\mathbf{r},\mathbf{r}')$ is symmetric in $\mathbf{r}$, $\mathbf{r}'$ (Walker 1942), the approach of § 5·1 is justified *a posteriori*. Equation (84) may be written

$$\rho_t(\mathbf{r}) = \operatorname*{average}_{\widehat{\mathbf{r}'\mathbf{r}}=\alpha} \rho_{t-1}(\mathbf{r}'), \tag{4}$$

the average according equal weight to patches $\mathrm{d}S'$ of $S(\mathbf{r},\alpha)$ that subtend equal solid angles at $\mathbf{r}$. The characteristic functions are solutions of

$$\operatorname*{average}_{\widehat{\mathbf{r}'\mathbf{r}}=\alpha} \phi_\lambda(\mathbf{r}') = \lambda\phi_\lambda(\mathbf{r}), \tag{5}$$

which may be rewritten

$$\overline{\phi_\lambda(\mathbf{r};\alpha)} \equiv \frac{1}{\Omega}\int_{S(\mathbf{r},\alpha)} \frac{\phi_\lambda(\mathbf{r}')\,\mathrm{d}S'}{\Phi(\mathbf{r},\mathbf{r}')} = \lambda\phi_\lambda(\mathbf{r}). \tag{85}$$

The integral of equation (85) may be expressed as an infinite series (see appendix B, equation (B 12)) which, for small α, reduces approximately to

$$\overline{\phi_\lambda(\mathbf{r};\alpha)} \approx \phi_\lambda(\mathbf{r}) + \frac{1}{2q}\alpha^2\nabla^2\phi_\lambda(\mathbf{r}). \tag{86}$$

† It may happen that $\tau_1(\mathbf{r},\mathbf{r}')$, for example, possesses a degeneracy $\lambda_{11} = \lambda_{21}$ and that the ϕ_{11}, ϕ_{21}, first chosen as base are not characteristic functions of (say) $\tau_2(\mathbf{r},\mathbf{r}')$. In such a case it may still be possible to obtain characteristic functions independent of t by replacing ϕ_{11}, ϕ_{21} by suitable linear combinations.

Thus, the characteristic functions for an infinitesimal step satisfy the diffusion equation

$$g^{ij}\phi_{(n),ij} \equiv \nabla^2\phi_n = \zeta_n\phi_n \tag{87}$$

approximately, the corresponding value of λ being

$$\lambda = 1+\frac{\alpha^2}{2q}\zeta_n, \tag{88}$$

to the order of accuracy warranted. A diffusion of variance V turns an initial distribution

$$\rho_0 = \Sigma C_n\phi_n$$

into

$$\rho = \Sigma \mathrm{e}^{V\zeta_n/2q}C_n\phi_n,$$

cf. equation (14). The characteristic functions ϕ_n in (87) are, by hypothesis, everywhere twice differentiable and so *a fortiori* bounded. We have used in equation (87) the summation convention in the indices i, j and the usual notation for covariant derivatives. In full

$$\phi_{,ij} \equiv \phi_{,i,j} \equiv \frac{\partial^2\phi}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial\phi}{\partial x^k}$$

$$\Gamma^k_{ij} = \tfrac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial x^j}+\frac{\partial g_{lj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^l}\right).$$

We suppose that the set ϕ_n is complete and orthonormal and bear in mind the possible existence of degeneracies.

For large α, the functions determined by the differential equation (87) may be expected to be solutions of the integral equation (85) for certain types of space, but not for all. We will show that one of these special types is the class of (completely) harmonic spaces, i.e. those† for which

$$\Phi(\mathbf{r},\mathbf{r}') = \Phi(\alpha). \tag{89}$$

When this holds, we find

$$\Omega\Phi(\alpha)\,\frac{\partial\overline{\phi_n(\mathbf{r};\,\alpha)}}{\partial\alpha} \equiv \int_{S(\mathbf{r},\,\alpha)}\frac{\partial\phi_n(\mathbf{r}')}{\partial n}\,\mathrm{d}S', \tag{90}$$

since the $\Phi(\alpha)$ may be taken outside the integration sign. Now, applying the generalized divergence theorem to the right-hand side of equation (90), we obtain

$$\Omega\Phi(\alpha)\,\frac{\partial\overline{\phi_n(\mathbf{r};\,\alpha)}}{\partial\alpha} \equiv \int_{V(\mathbf{r},\,\alpha)}\nabla^2\phi_n(\mathbf{r}')\,\mathrm{d}V' = \zeta_n\int_{V(\mathbf{r},\,\alpha)}\phi_n(\mathbf{r}')\,\mathrm{d}V'. \tag{91}$$

Differentiating this result with respect to α and bringing $\Phi(\alpha)$ through the integration sign again, we find

$$\Omega\frac{\partial}{\partial\alpha}\left(\Phi(\alpha)\,\frac{\partial\overline{\phi_n(\mathbf{r};\,\alpha)}}{\partial\alpha}\right) = \zeta_n\Phi(\alpha)\int_{S(\mathbf{r},\,\alpha)}\frac{\phi_n(\mathbf{r}')\,\mathrm{d}S'}{\Phi(\mathbf{r},\mathbf{r}')}. \tag{92}$$

Thus, *provided we treat* $\mathbf{r}$ *as constant*, $\overline{\phi_n(\mathbf{r};\,\alpha)}$ satisfies the ordinary differential equation

$$\frac{1}{\Phi}\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\Phi\frac{\mathrm{d}\psi_n}{\mathrm{d}\alpha}\right) = \zeta_n\psi_n. \tag{93}$$

† The definitions that have been given for harmonic spaces have not been very decided on the range of α for which equation (89) is to hold. Clearly, if the space is analytic, the question does not arise. We shall require it to hold for all α up to the length of the longest random step.

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It follows that, at least for each characteristic value ζ_n of (87), equation (93) possesses a solution which is bounded whenever equation (89) holds. Now, since

$$\Phi(\alpha) \sim \alpha^{q-1} \quad \text{as} \quad \alpha \to 0, \tag{94}$$

equation (93) has a singularity at $\alpha = 0$, and there is only one solution which remains bounded (apart from the arbitrary constant factor). This solution is non-zero at $\alpha = 0$. Denote it by $\psi_n(\alpha)$ and fix the arbitrary constant factor by the condition

$$\psi_n(0) = 1. \tag{95}$$

It follows that $\overline{\phi_n(\mathbf{r};\alpha)}$, considered as a function of α for fixed $\mathbf{r}$, is a multiple of $\psi_n(\alpha)$, whence

$$\overline{\phi_n(\mathbf{r};\alpha)} = \psi_n(\alpha)\,\phi_n(\mathbf{r}). \tag{96}$$

Thus, in harmonic spaces, the characteristic functions of equation (87) are characteristic functions of equation (85) for all α, the corresponding characteristic values being $\psi_n(\alpha)$ or, for problems with the third extension,

$$\int_\alpha \psi_n(\alpha)\,\mathrm{d}p(\alpha).$$

Equation (82) therefore holds and there is a complete analogy with the formal solution of § 2.

(We may note in passing that equation (96) may be regarded as including Willmore's theorem (1950) on mean values over geodesic spheres of harmonic functions in harmonic spaces. See also Walker 1945*a*, 1947.)

5·3. We present here some interesting consequences of this analysis which are, however, irrelevant to the subsequent development.

In the first place, we observe that equation (93) may be written

$$\nabla^2\psi_n = \zeta_n\psi_n. \tag{97}$$

Suppose, in fact, that the geodesic distance $\alpha = \widehat{\mathbf{r}'\mathbf{r}}$ is a single-valued function of the position $\mathbf{r}'$ in a certain (small) neighbourhood of $\mathbf{r}$, and consider a function $\psi(\alpha)$ of this distance as a function of $\mathbf{r}'$ in the neighbourhood. We may apply the generalized divergence theorem to the volume V of a narrow cone of geodesics radiating from $\mathbf{r}$ to the area $\mathrm{d}S'$ at $\mathbf{r}'$ on $S(\mathbf{r},\alpha)$

$$\int_V \nabla^2\psi\,\mathrm{d}V = \int \frac{\partial\psi}{\partial n}\,\mathrm{d}S = \int \frac{\mathrm{d}\psi}{\mathrm{d}\alpha}\,\mathrm{d}S'; \tag{98}$$

the integration over the remaining surface of the cone clearly makes no contribution. But we may rewrite the right-hand side of this equation

$$\int_{\mathrm{d}S'} \frac{\mathrm{d}\psi}{\mathrm{d}\alpha}\,\mathrm{d}S' = \int \Phi\frac{\mathrm{d}\psi}{\mathrm{d}\alpha}\,\mathrm{d}\omega = \int \frac{\partial}{\partial\alpha}\left(\Phi\frac{\mathrm{d}\psi}{\mathrm{d}\alpha}\right)\mathrm{d}\alpha\,\mathrm{d}\omega = \int_V \frac{1}{\Phi}\frac{\partial}{\partial\alpha}\left(\Phi\frac{\mathrm{d}\psi}{\mathrm{d}\alpha}\right)\mathrm{d}V. \tag{99}$$

Thus, it follows that

$$\int_V \left[\nabla^2\psi - \frac{1}{\Phi}\frac{\partial}{\partial\alpha}\left(\Phi\frac{\mathrm{d}\psi}{\mathrm{d}\alpha}\right)\right]\mathrm{d}V = 0. \tag{100}$$

Differentiating this result with respect to α and shrinking the area $\mathrm{d}S'$ to zero, we find

$$\nabla^2\psi(\alpha) \equiv \frac{1}{\Phi}\frac{\partial}{\partial\alpha}\left(\Phi\frac{\mathrm{d}\psi(\alpha)}{\mathrm{d}\alpha}\right); \tag{101}$$

a result independent of any assumption about Φ. This establishes the equivalence of equations (93) and (97). Equation (101) was obtained by Ruse (1930*a*) by a different method.

Suppose next that equation (89) holds, i.e. the space is harmonic. Then, for any ζ_n, equation (87) has a solution which is a function of geodesic distance $\alpha = \widehat{\mathbf{r}_0\mathbf{r}}$ from the fixed point $\mathbf{r}_0$ only, and which is well defined and continuous in a (small) neighbourhood of $\mathbf{r}_0$: namely, the solution to equation (93) which is continuous at $\alpha = 0$.

It does not follow that we obtain in this way a solution of equation (87) which is single-valued and continuous over the whole manifold. In general, two points $\mathbf{r}$, $\mathbf{r}_0$ can be joined by an infinity of geodesic arcs of different lengths $\alpha_1, \alpha_2, \ldots$.

If the space is *globally* harmonic, in the sense that equation (89) holds for geodesic arcs of arbitrary length α, and if ζ_n is a characteristic value of equation (87), then equation (93) has a solution $\psi_n(\alpha)$ which is *everywhere* finite. This fact follows from equation (96), applied with a point at which $\phi_n(\mathbf{r}_0)$ is not zero. The remarkable feature of this result is that equation (93) has a singularity at each of the zeros of $\Phi(\alpha)$, but the same solution of (93) is continuous at all these singularities of the differential equation.

The functions $\psi_n(\alpha)$ may have the property that

$$\psi_n(\alpha_1) = \psi_n(\alpha_2) = \ldots, \tag{102}$$

for all the geodesic distances $\alpha_1, \alpha_2, \ldots$ between any two points $\mathbf{r}$, $\mathbf{r}_0$. In this case—and only in this case—$\psi_n(\alpha)$ gives a solution of (87) which is single-valued and continuous over the whole manifold and is a function of 'the' geodesic distance from a fixed point $\mathbf{r}_0$.

While all the conditions (102) are necessary for this purpose, it is, in fact, sufficient that

$$\psi_n(\alpha_1) = \psi_n(\alpha_2), \tag{103}$$

for the two shortest geodesic distances. To see this, define

$$\xi_n(\mathbf{r}) = \psi_n(\alpha_1), \tag{104}$$

where α_1 is the shortest geodesic distance from a fixed point $\mathbf{r}_0$ to $\mathbf{r}$. Along any geodesic from $\mathbf{r}_0$, there is an initial arc in which this is, in fact, the shortest geodesic. Let $\mathbf{r}_1$ be the endpoint of this arc. On the arc $\widehat{\mathbf{r}_0\mathbf{r}_1}$, $\xi_n(\mathbf{r})$ is the same as $\psi_n(\alpha)$, α being the arc length along this particular geodesic, and hence $\xi_n(\mathbf{r})$ satisfies equation (87). In the neighbourhood of $\mathbf{r}_1$, there are points for which $\alpha = \alpha_2$, but, by (103), it is still true that $\xi_n(\mathbf{r}) = \psi_n(\alpha)$. Hence $\xi_n(\mathbf{r})$ satisfies equation (87) also at $\mathbf{r}_1$. Thus, $\xi_n(\mathbf{r})$ satisfies equation (87) everywhere. It follows, by analytic continuation, that it equals $\psi_n(\alpha)$ for all geodesic arcs $\widehat{\mathbf{r}_0\mathbf{r}}$. Thus, equation (102) follows.

For fixed $\mathbf{r}_0$, the various geodesic arcs from $\mathbf{r}_0$ give points $\mathbf{r}_1$ which describe a submanifold $\mathscr{S} = \mathscr{S}(\mathbf{r}_0)$. It is sometimes called the cut-locus; cf. Whitehead (1935, p. 700) where further references may be found. On our present hypotheses—summarized by equations (89) and (103)—this submanifold $\mathscr{S}(\mathbf{r}_0)$ is in fact a geodesic sphere of centre $\mathbf{r}_0$.

To see this we suppose first that

$$\Phi(\alpha) \neq 0 \quad \text{at} \quad \mathbf{r}_1. \tag{105}$$

Then there must be two distinct geodesic arcs γ', γ'' of equal length from $\mathbf{r}_0$ to $\mathbf{r}_1$. Since the arcs are distinct, their directions at $\mathbf{r}_1$ must be distinct, though they could be (and in fact are) directly opposed. By (105), affine normal co-ordinates (cf. Veblen 1927, chap. 6) with centre $\mathbf{r}_0$ and derived from the sheaf of geodesics near to γ' form a proper co-ordinate system near $\mathbf{r}_1$. We denote by α' the arc length from $\mathbf{r}_0$ to any point near $\mathbf{r}_1$ along the geodesic arc in this sheaf. Similarly, we define α'' from the sheaf of geodesics near γ''. Since the directions of γ', γ'' are distinct, the gradients of the functions α', α'' at $\mathbf{r}_1$ are different (although they may be, and in fact are, equal and opposite). The submanifold $\mathscr{S}(\mathbf{r}_0)$ is given by the equation

$$\alpha' = \alpha'',$$

which shows that it is in fact $(q-1)$-dimensional, and that it makes the same angle θ with γ' and γ'', which approach it from opposite sides. We now use (103) for points just off $\mathscr{S}(\mathbf{r}_0)$ and obtain

$$\psi_n'(\alpha)\sin\theta = 0.$$

Here $\sin\theta \neq 0$, since γ', γ'' have distinct directions at $\mathbf{r}_1$, and so

$$\psi_n'(\alpha) = 0, \tag{106}$$

which fixes α. By continuity, the radius $\mathscr{s}$ of $\mathscr{S}(\mathbf{r}_0)$, the appropriate solution of (106), is independent of $\mathbf{r}_0$. Since α is constant on $\mathscr{S}(\mathbf{r}_0)$ it follows that $\theta = \frac{1}{2}\pi$ and so that γ'' is the continuation of γ' in reverse. Thus after an arc length $2\mathscr{s}$ every geodesic returns to its starting point: in particular $\Phi(2\mathscr{s}) = 0$. An example of this is the real projective space, represented as a unit sphere with diametrically opposite points identified, when

$$\Phi(\alpha) = \sin^{q-1}\alpha, \quad \mathscr{s} = \tfrac{1}{2}\pi.$$

On the other hand, suppose that

$$\Phi(\alpha) = 0 \quad \text{at} \quad \mathbf{r}_1. \tag{107}$$

Then α is fixed by (107), so that again $\mathscr{S}(\mathbf{r}_0)$ is a geodesic sphere of centre $\mathbf{r}_0$ and radius $\mathscr{s}$ independent of $\mathbf{r}_0$. In this case, however, $\mathscr{S}(\mathbf{r}_0)$ is of dimension only $q-r-1$, where r is the order† of the zero in (107). In this case also the prolongation of one geodesic from $\mathbf{r}_0$ beyond the point $\mathbf{r}_1$ in which it meets $\mathscr{S}(\mathbf{r}_0)$ is simply another geodesic from $\mathbf{r}_0$ to $\mathbf{r}_1$ in reverse, so that every geodesic ray returns to its starting point after an arc length $2\mathscr{s}$.‡

We may now establish an addition theorem for the function $\psi_n(\widehat{\mathbf{r}_0\mathbf{r}})$. Define Ψ_n by

$$\Psi_n = \int_M [\psi_n(\widehat{\mathbf{r}_0\mathbf{r}})]^2\,\mathrm{d}V = \Omega\int_0^{\mathscr{s}} [\psi_n(\alpha)]^2\Phi(\alpha)\,\mathrm{d}\alpha. \tag{108}$$

It is convenient to normalize $\xi_n(\mathbf{r})$: define

$$\phi_{n0}(\mathbf{r}) = \frac{1}{\sqrt{\Psi_n}}\xi_n(\mathbf{r}). \tag{109}$$

† The extreme values of r occur for the sphere ($r = q-1$) and the complex projective plane ($r = 1$). For the latter cf. the footnote to §5·4.

‡ If a harmonic space be such that $\Phi(\alpha_0) = 0$ for some $\alpha_0 > 0$, then every geodesic ray returns to its starting point after an arc length $2\alpha_0$. For the geodesic sphere $S(\mathbf{r}_0, \alpha_0)$ is of zero $(q-1)$-dimensional measure, therefore of dimension less than $(q-1)$, say $(q-r-1)$ where $r \geqslant 1$ can in fact be shown to be the order of the zero of Φ at α_0. Each point $\mathbf{r}_1$ on $S(\mathbf{r}_0, \alpha_0)$ is therefore the terminus of an r-dimensional pencil of geodesic rays from $\mathbf{r}_0$: these rays all meet the geodesic sphere $S(\mathbf{r}_0, \alpha_0)$ at right angles: and they fall therefore into pairs which arrive at $\mathbf{r}_1$ from directly opposed directions. Hence the prolongation beyond $\mathbf{r}_1$ of one member of such a pair is simply the other member of that pair in reverse.

It would follow that $$(-1)^{q-1}\Phi(-\alpha) = \Phi(\alpha) = (-1)^r\Phi(2\alpha_0-\alpha).$$

Thus $\Phi(\alpha)$ would be periodic of period $4\alpha_0$: apart from a possible sign it would be periodic of period $2\alpha_0$. All the functions $\psi_n(\alpha)$ would be even functions of period $2\alpha_0$. It would also follow that a topological sphere of $(q-1)$ dimensions could be expressed as a fibre space (Steenrod 1951) with r-dimensional spheres as the fibres: few examples of this are known to us.

Let $\phi_{ni}(\mathbf{r})$ $(i = 1, 2, \ldots)$ be other characteristic functions which, together with $\phi_{n0}(\mathbf{r})$, form an orthonormal base for solutions of equation (87). Consider the integral

$$\int_M \psi_n(\widehat{\mathbf{r}'\mathbf{r}})\, \phi_{ni}(\mathbf{r}')\, \mathrm{d}V'.$$

This may be evaluated by integrating first over $S(\mathbf{r}, \alpha')$ and then over α' and by making use of equation (96):

$$\int_M \psi_n(\widehat{\mathbf{r}'\mathbf{r}})\, \phi_{ni}(\mathbf{r}')\, \mathrm{d}V' = \int_0^\delta \mathrm{d}\alpha' \psi_n(\alpha')\, \Phi(\alpha') \int_{S(\mathbf{r},\,\alpha)} \frac{\phi_{ni}(\mathbf{r}')\, \mathrm{d}S'}{\Phi(\mathbf{r}, \mathbf{r}')}$$
$$= \Psi_n \phi_{ni}(\mathbf{r}). \tag{110}$$

This establishes the addition theorem for $\psi_n(\widehat{\mathbf{r}'\mathbf{r}})$:

$$\psi_n(\widehat{\mathbf{r}'\mathbf{r}}) = \Psi_n \sum_i \phi_{ni}(\mathbf{r}')\, \phi_{ni}(\mathbf{r}). \tag{111}$$

In passing, it is interesting to note that, since the left-hand side of equation (110) vanishes $(i \neq 0)$ at the point $\mathbf{r} = \mathbf{r}_0$ from the orthogonality of the base chosen, it follows that

$$\phi_{ni}(\mathbf{r}_0) = 0 \quad (i > 0). \tag{112}$$

This completes the analogy with the working of § 2; cf. the proof of equation (6).

5·4. We have shown in § 5·2 that the method of § 2 for a sphere will generalize at once to closed harmonic spaces. This is a very restricted class of manifold but is more general than that of closed spaces of constant curvature. Willmore (1953) has given several examples of closed harmonic spaces not of constant curvature. Walker (1945 *b*, § 3, equation (10)) first constructed a harmonic space not of constant curvature. As defined by Walker, the space is not a closed manifold, but it can be closed by the addition† of a two-dimensional manifold 'at infinity'. Further, it satisfies the conditions necessary for the validity of the addition theorem (111). We now inquire what are the conditions on the space in order that our method shall apply for *every* random walk, and in particular whether there exist any other spaces besides harmonic spaces which have the required property.

A space with this property we shall call a *commutative space*. We recall that, by definition, the property in question is that there exist a complete orthonormal set of functions each of which is a characteristic function for every random step. This is clearly equivalent to the property that *any two random steps commute*. Every harmonic space is commutative.

Each ϕ_n in the supposed set of common characteristic functions must be a characteristic function in particular for infinitesimal steps, and so satisfies (87) with some value of ζ_n. Thus the common characteristic functions are determined as those of the operator ∇^2—apart from the very real possibility of degeneracy among the latter.

† In fact, the substitution
$$w_1/(x - \mathrm{i}y) = w_2/(z + \mathrm{i}t) = w_3(\tfrac{1}{2}K)^{\frac{1}{2}}$$
reduces Walker's manifold to the (complex) projective 'plane', with homogeneous co-ordinates (w_1, w_2, w_3) not all zero, endowed with the metric
$$\mathrm{d}s^2 = \frac{|w_2\,\mathrm{d}w_3 - w_3\,\mathrm{d}w_2|^2 + |w_3\,\mathrm{d}w_1 - w_1\,\mathrm{d}w_3|^2 + |w_1\,\mathrm{d}w_2 - w_2\,\mathrm{d}w_1|^2}{\tfrac{1}{2}K(|w_1|^2 + |w_2|^2 + |w_3|^2)^2},$$
the extra points added 'at infinity' being those of the 'line' $w_3 = 0$. This metric is clearly invariant under unitary transformations of the variables w_i, so that the space possesses a transitive group of motions and is therefore symmetric.

Now we can write

$$\tau_t(\mathbf{r},\mathbf{r}') = \sum_m \sum_n a_{mn}\phi_m(\mathbf{r})\,\phi_n(\mathbf{r}'), \tag{113}$$

where the a_{mn} may of course depend also on t. Applying this in (75) with

$$\rho_{t-1}(\mathbf{r}') = \phi_k(\mathbf{r}')$$

we get

$$\rho_t(\mathbf{r}) = \sum_m a_{mk}\phi_m(\mathbf{r}).$$

Thus in order that the ϕ_n be in fact common characteristic functions for every random step it is necessary and sufficient that, for *every* random step,

$$\text{the matrix } [a_{mn}] \text{ is diagonal.} \tag{114}$$

It is therefore necessary that, for every random step,

$$\nabla^2_{(\mathbf{r})}\tau(\mathbf{r},\mathbf{r}') = \nabla^2_{(\mathbf{r}')}\,\tau(\mathbf{r},\mathbf{r}'). \tag{115}$$

This condition would be also sufficient for commutativity if the characteristic value problem (87) were free from degeneracies. But degeneracies are only to be expected, and so (115) only reduces the matrices $[a_{mn}]$ to diagonal block form, the different blocks corresponding to the different characteristic values ζ_n of the operator ∇^2. Any one such matrix may of course be reduced to diagonal form by a change of basis, but commutativity requires a basis $\{\phi_n\}$ which reduces all the matrices $[a_{mn}]$ simultaneously to diagonal form. Thus (115) is necessary but not, so far as we know, sufficient for commutativity.

Condition (115) may also be derived in the following alternative way. If arbitrary random steps commute, then any random step must commute with an infinitesimal step, i.e. with diffusion. According to equations (75) and (86), this requires that

$$\nabla^2_{(\mathbf{r})}\int_M \tau(\mathbf{r},\mathbf{r}')\,\rho(\mathbf{r}')\,\mathrm{d}V' = \int_M \tau(\mathbf{r},\mathbf{r}')\,\nabla^2_{(\mathbf{r}')}\rho(\mathbf{r}')\,\mathrm{d}V', \tag{116}$$

and this, by an application of the generalized Green's theorem, is equivalent to

$$\int_M \rho(\mathbf{r}')\,[\nabla^2_{(\mathbf{r})}-\nabla^2_{(\mathbf{r}')}]\,\tau(\mathbf{r},\mathbf{r}')\,\mathrm{d}V' = 0. \tag{117}$$

Equation (117) must hold true for all $\rho(\mathbf{r})$, and thus equation (115) is recovered.

Condition (115) requires 'interpretation' in general, and in particular for the basic case of a random step of fixed length α. Its meaning is clear in the case of a random step given by a cumulative probability distribution $p(\alpha)$ which is thrice differentiable; for then, by the arguments with which § 5·2 began,

$$\tau(\mathbf{r},\mathbf{r}') = \frac{1}{\Omega\Phi(\mathbf{r},\mathbf{r}')}\,\frac{\mathrm{d}p(\alpha)}{\mathrm{d}\alpha}, \tag{118}$$

and, using equation (101),

$$\Omega\nabla^2_{(\mathbf{r})}\tau(\mathbf{r},\mathbf{r}') = \frac{1}{\Phi(\mathbf{r},\mathbf{r}')}\,\frac{\mathrm{d}^3p(\alpha)}{\mathrm{d}\alpha^3} + g^{ij}(\mathbf{r})\,\frac{\partial\alpha}{\partial x^i}\,\frac{\partial}{\partial x^j}\left[\frac{1}{\Phi(\mathbf{r},\mathbf{r}')}\right]\frac{\mathrm{d}^2p(\alpha)}{\mathrm{d}\alpha^2} + \nabla^2_{(\mathbf{r})}\left[\frac{1}{\Phi(\mathbf{r},\mathbf{r}')}\right]\frac{\mathrm{d}p(\alpha)}{\mathrm{d}\alpha}. \tag{119}$$

Thus, (115) gives rise to the two geometrical conditions

$$g^{ij}(\mathbf{r})\frac{\partial\alpha}{\partial x^i}\frac{\partial}{\partial x^j}\left[\frac{1}{\Phi(\mathbf{r},\mathbf{r}')}\right] = g^{ij}(\mathbf{r}')\frac{\partial\alpha}{\partial x^{i'}}\frac{\partial}{\partial x^{j'}}\left[\frac{1}{\Phi(\mathbf{r},\mathbf{r}')}\right], \tag{120}$$

$$\nabla^2_{(\mathbf{r})}\left[\frac{1}{\Phi(\mathbf{r},\mathbf{r}')}\right] = \nabla^2_{(\mathbf{r}')}\left[\frac{1}{\Phi(\mathbf{r},\mathbf{r}')}\right]. \tag{121}$$

The first of these is similar to (89), but whereas equation (89) requires that Φ is a function of geodesic separation alone, condition (120) only requires that, along any *particular* geodesic, Φ is a function of α alone. Clearly, this does not prevent that function from varying from geodesic to geodesic in any manner consistent with equation (121).

The normal tensors can be used to bring out the differences between conditions (89) and (120). Let $\mathbf{r}$ be the base point of a system of *affine normal co-ordinates* in which the co-ordinates of $\mathbf{r}'$ are y^i. The components of the affine extension may be expanded in a Taylor series (cf. Veblen 1927, chap. 6, equation (6·4)),

$$\Gamma^i_{jk}(\mathbf{r}') = A^i_{jka}(\mathbf{r})\,y^a + \frac{1}{2!}A^i_{jkab}(\mathbf{r})\,y^a y^b + \frac{1}{3!}A^i_{jabc}(\mathbf{r})\,y^a y^b y^c + \dots, \tag{122}$$

where $A^i_{jka}(\mathbf{r})$, etc., are the components of the *affine normal tensors* in the co-ordinate system y^i.

Since
$$\Gamma^i_{ik}(\mathbf{y}) \equiv \frac{\partial}{\partial y^k}\ln\sqrt{g(\mathbf{y})} = \frac{\partial}{\partial y^k}\ln\frac{\Phi(\mathbf{r},\mathbf{r}')}{\alpha^{n-1}}, \tag{123}$$

we have
$$\ln\frac{\Phi(\mathbf{r},\mathbf{r}')}{\alpha^{n-1}} = \frac{1}{2!}A^\alpha_{\alpha ij}(\mathbf{r})\,y^i y^j + \frac{1}{3!}A^\alpha_{\alpha ijk}(\mathbf{r})\,y^i y^j y^k + \frac{1}{4!}A^\alpha_{\alpha ijkl}(\mathbf{r})\,y^i y^j y^k y^l \dots. \tag{124}$$

Condition (89) for a harmonic space requires that each individual term in this series is a function of α alone, whence:

$$\left.\begin{aligned}(S_{ij})\,A^\alpha_{\alpha ij} &= k_1(S_{ij})\,g_{ij},\\ S_{ijkl}A^\alpha_{\alpha ijkl} &= k_2 S_{ijkl}\,g_{ij}\,g_{kl},\\ S_{ijklmn}A^\alpha_{\alpha ijklmn} &= k_3 S_{ijklmn}\,g_{ij}\,g_{kl}\,g_{mn},\\ &\dots\dots\dots\dots\\ &(k_1, k_2, k_3, \dots \text{constants}).\end{aligned}\right\} \tag{125a}$$

$$\left.\begin{aligned}(S_{ijk})\,A^\alpha_{\alpha ijk} &= 0,\\ S_{ijklm}A^\alpha_{\alpha ijklm} &= 0,\\ S_{ijklmno}A^\alpha_{\alpha ijklmno} &= 0,\\ &\dots\dots\dots\end{aligned}\right\} \tag{125b}$$

where $S_{ijk}\dots$ denotes the symmetric part in the suffixes $i, j, k, \dots$ of the expression following it. (It may be omitted when enclosed in brackets.) This infinite set of conditions has been postulated by Copson & Ruse (1940, equations (7·19) to (7·23)). Condition (120) for a commutative space requires that the series (124) is unchanged by replacing y^i by $-y^i$ everywhere. Thus, conditions (125b) must still be satisfied and the first of these establishes that all two-dimensional commutative spaces are of constant curvature (cf. Copson & Ruse 1940, equations (7·16) et seq.). Conditions (125a), on the other hand, need not be satisfied by commutative spaces.

In appendix B we introduce a sequence of invariant linear differential operators $\Delta_{2\nu}$ ($\nu = 1, 2, \ldots$), of the order indicated by the suffix. If both the metric g_{ij} and the distribution $\rho(\mathbf{r})$ are analytic we obtain

$$\overline{\rho(\mathbf{r};\alpha)} = \sum_{\nu=0}^{\infty} \frac{\Gamma(\frac{1}{2}q)}{\nu!\,\Gamma(\nu+\frac{1}{2}q)} (\tfrac{1}{4}\alpha^2)^\nu \Delta_{2\nu}\rho(\mathbf{r}), \tag{126}$$

for sufficiently small α. This leads to the conditions

$$\Delta_{2\mu}\Delta_{2\nu} \equiv \Delta_{2\nu}\Delta_{2\mu} \tag{127}$$

as necessary conditions for commutativity in an analytic space.

If the space is analytic then the conditions (127) are sufficient as well as necessary for commutativity. Their sufficiency follows first for distributions $\rho(\mathbf{r})$ at least locally analytic, and for steps of lengths α, β sufficiently small, but the result can be freed from these restrictions (see appendix C). The conditions (115) are equivalent to the conditions (127) with $\mu = 1$.

It is not known whether any commutative manifold exists which is not analytic, but if so then it must satisfy those conditions (127) which on it make sense.

The question whether there exist commutative spaces which are not harmonic is answered by the following theorems:

(I) Every harmonic space is commutative.

(II) The product of two harmonic spaces is not in general harmonic.

(III) The product of two commutative spaces is commutative.

Of these theorems, (I) has already been established, and we shall prove (II) and (III) in § 5·5. In particular, the product of a circle and a sphere, a three-dimensional manifold which can be embedded in ordinary five-dimensional space, is commutative but not harmonic.

5·5. Given two spaces M_1 and M_2, of typical points $\mathbf{r}_1$ and $\mathbf{r}_2$, the product space $M = M_1 \times M_2$ is defined to consist of points $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2)$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ range independently over M_1 and M_2. If M_1 and M_2 are Riemannian manifolds with metrics $\mathrm{d}s_1^2$ and $\mathrm{d}s_2^2$, then it is understood that M is assigned the metric

$$\mathrm{d}s^2 = \mathrm{d}s_1^2 + \mathrm{d}s_2^2. \tag{128}$$

A geodesic arc in M is given by two geodesic arcs in M_1 and M_2, with the arc lengths related by

$$\mathrm{d}s_1 = \mathrm{d}s\cos\theta, \quad \mathrm{d}s_2 = \mathrm{d}s\sin\theta, \quad \theta = \text{const.} \tag{129}$$

This follows for short arcs from the minimal property of a geodesic (and elementary plane geometry): verification from the geodesic equations is immediate. Now consider a thin sheaf of geodesic arcs, all of length α, starting from a common origin in M and occupying at that origin a small solid angle $\mathrm{d}\omega$ formed by

a small solid angle $\mathrm{d}\omega_1$ in M_1,
a small solid angle $\mathrm{d}\omega_2$ in M_2,
a small range $\mathrm{d}\theta$ of θ.

Let Φ refer to the arc in M and let Φ_1, Φ_2 refer to the corresponding arcs in M_1, M_2, whose lengths we denote by α_1, α_2. Then

$$\alpha_1 = \alpha\cos\theta, \quad \alpha_2 = \alpha\sin\theta, \quad \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2 = \alpha\,\mathrm{d}\alpha\,\mathrm{d}\theta$$

Also the volume element in M is

$$\Phi\,\mathrm{d}\omega\,\mathrm{d}\alpha = \Phi_1\,\mathrm{d}\omega_1\,\mathrm{d}\alpha_1\,\Phi_2\,\mathrm{d}\omega_2\,\mathrm{d}\alpha_2.$$

Hence
$$\Phi\,\mathrm{d}\omega = \alpha\Phi_1\Phi_2\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\theta. \tag{130}$$

Making $\alpha \to 0$ we have $\quad \Phi \sim \alpha^{q-1}, \quad \Phi_1 \sim \alpha_1^{q_1-1}, \quad \Phi_2 \sim \alpha_2^{q_2-1},$

where q_1, q_2 are the dimensions of M_1, M_2 and where

$$q = q_1 + q_2$$

is the dimension of M. From equation (130) we therefore get

$$\mathrm{d}\omega = \cos^{q_1-1}\theta\,\sin^{q_2-1}\theta\,\mathrm{d}\omega_1\,\mathrm{d}\omega_2\,\mathrm{d}\theta, \tag{131}$$

and so
$$\Phi = \alpha\Phi_1\Phi_2\sec^{q_1-1}\theta\,\operatorname{cosec}^{q_2-1}\theta. \tag{132}$$

Equation (132) takes simpler form in terms of Ruse's invariant ρ (Walker 1942): if we define

$$\rho = \Phi\alpha^{1-q}, \quad \rho_1 = \Phi_1\alpha_1^{1-q_1}, \quad \rho_2 = \Phi_2\alpha_2^{1-q_2},$$

then (132) becomes
$$\rho = \rho_1\rho_2. \tag{133}$$

Now suppose that M_1 and M_2 are harmonic spaces, so that ρ_1, like Φ_1, is a function of α_1 alone, and that ρ_2 is a function of α_2 alone. In order that M also shall be harmonic, ρ must be a function of α alone, and these functions must satisfy

$$\rho(\alpha) = \rho_1(\alpha\cos\theta)\,\rho_2(\alpha\sin\theta),$$

for all values of α and θ. Varying θ with α constant we get

$$\rho_1'(\alpha_1)/\alpha_1\rho_1(\alpha_1) = \rho_2'(\alpha_2)/\alpha_2\rho_2(\alpha_2).$$

Since here α_1 and α_2 are independently variable, it follows that the two sides are equal constants, say $2c$, whence

$$\rho(\alpha) = \rho_1(\alpha) = \rho_2(\alpha) = \mathrm{e}^{c\alpha^2},$$

or, in terms of Φ,
$$\Phi_1(\alpha) = \alpha^{q_1-1}\,\mathrm{e}^{c\alpha^2}, \quad \Phi_2(\alpha) = \alpha^{q_2-1}\,\mathrm{e}^{c\alpha^2}. \tag{134}$$

Whether any closed harmonic space exists for which Φ takes the form (134)—except for simply harmonic spaces, $c = 0$—may be doubted: there is no doubt that harmonic spaces exist, e.g. spheres, which do not satisfy (134): and so (II) is established.

On the other hand, suppose that M_1 and M_2 are commutative spaces. Then in M_1 there is a complete orthonormal set of functions, each of which is a common characteristic function for every random step on M_1. Let $\psi_{m1}(\alpha_1)$ be the characteristic value corresponding to ϕ_{m1} when the random step is of fixed length α_1. Then

$$\overline{\phi_{m1}(\mathbf{r}_1;\,\alpha_1)} = \psi_{m1}(\alpha_1)\,\phi_{m1}(\mathbf{r}_1), \tag{135·1}$$

which corresponds exactly to equation (96). So also the manifold M_2 gives us a complete orthonormal set $\phi_{n2}(\mathbf{r}_2)$, and corresponding functions $\psi_{n2}(\alpha_2)$, such that

$$\overline{\phi_{n2}(\mathbf{r}_2;\,\alpha_2)} = \psi_{n2}(\alpha_2)\,\phi_{n2}(\mathbf{r}_2). \tag{135·2}$$

Now define on M the functions

$$\phi_{mn}(\mathbf{r}_1, \mathbf{r}_2) = \phi_{m1}(\mathbf{r}_1)\,\phi_{n2}(\mathbf{r}_2). \tag{136}$$

Evidently they form a complete orthonormal set on M. And from (131) we get

$$\overline{\phi_{mn}(\mathbf{r};\alpha)} = \psi_{mn}(\alpha)\,\phi_{mn}(\mathbf{r}), \tag{137}$$

where
$$\Omega\psi_{mn}(\alpha) = \Omega_1\Omega_2\int_0^{\frac{1}{2}\pi}\cos^{q_1-1}\theta\,\sin^{q_2-1}\theta\psi_{m1}(\alpha\cos\theta)\,\psi_{n2}(\alpha\sin\theta)\,\mathrm{d}\theta. \tag{138}$$

Here Ω, Ω_1, Ω_2 denote the total solid angles in spaces of dimensions q, q_1, q_2, respectively. This shows that $\phi_{mn}(r)$ is a characteristic function for a random step on M of any length α. For a random step on M with the third extension, it is still a characteristic function, the characteristic value being

$$\int \psi_{mn}(\alpha)\,\mathrm{d}p(\alpha).$$

Thus M is commutative and (III) is established.

It is worth noting that these results depend on the relation

$$\Omega R(\alpha) = \Omega_1\Omega_2\int_0^{\frac{1}{2}\pi}\cos^{q_1-1}\theta\,\sin^{q_2-1}\theta\,R_1(\alpha\cos\theta)\,R_2(\alpha\sin\theta)\,\mathrm{d}\theta. \tag{139}$$

Here $R_1(\alpha_1)$ is an *operator* acting in the space M_1. It is in fact simply the operation of a random step of length α_1 in that space. Similarly $R_2(\alpha_2)$ is the operation of a random step of length α_2 in M_2 and $R(\alpha)$ the operation of a random step of length α in M.

Finally, we give another proof of (III) depending on the criterion (127). This proof assumes that M_1 and M_2 are *analytic* spaces: and it also assumes the *sufficiency* of (127), which is discussed in appendix C. We observe, then, that we have linear differential operators $\Delta_{2\mu,1}$ in the space M_1. These involve differentiations with respect to the co-ordinates on M_1, and also functions of those co-ordinates, derived from the metric tensor on M_1: but it does not involve either functions of the co-ordinates on M_2 nor differentiations with respect to them. So also, *mutatis mutandis*, for the linear differential operators $\Delta_{2\nu,2}$ on M_2. Both sets of operators can be interpreted as operators on M, and in view of what has been said it is clear that

$$\Lambda_{2\mu,1} \quad \text{commutes with} \quad \Delta_{2\nu,2}. \tag{140}$$

Since M_1 is commutative, the operators $\Delta_{2\mu,1}$ commute among themselves: since M_2 is commutative, the operators $\Delta_{2\nu,2}$ commute among themselves. Thus

$$\text{the operators } \Delta_{2\mu,1} \text{ and } \Delta_{2\nu,2} \text{ all commute.} \tag{141}$$

From what we said earlier about geodesics on M, it is clear that a set of affine normal co-ordinates on M at a point $(\mathbf{r}_1, \mathbf{r}_2)$ can be made up of a set of affine normal co-ordinates on M_1 at $\mathbf{r}_1$ together with a set of affine normal co-ordinates on M_2 at $\mathbf{r}_2$. In particular, if we choose pseudo-Euclidean co-ordinates (cf. appendix B, following equation (B 5)) on M_1 at $\mathbf{r}_1$ and on M_2 at $\mathbf{r}_2$, the two sets together give pseudo-Euclidean co-ordinates on M at $(\mathbf{r}_1, \mathbf{r}_2)$. Referring to appendix B, equation (B 15), we see that

$$\Delta_{2\lambda} = \sum_{\mu+\nu=\lambda}\frac{\lambda!}{\mu!\,\nu!}\Delta_{2\mu,1}\Delta_{2\nu,2}. \tag{142}$$

From (141) and (142), it is clear that the operators $\Delta_{2\lambda}$ on M commute: and so, by the sufficiency of (127), that M is commutative.

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5·6. In this final section we offer a few remarks on the possibility of generalizing the work of §§ 5·1 to 5·5 to manifolds that are not compact.

There is little formal change in § 5·1. Now, however, the spectrum of (75) will include a continuous part; thus, integrals will appear in place of summations. Neglecting a possible discrete part, the characteristic functions for the tth step will depend on continuous parameters ($\boldsymbol{\xi}$, say) and will satisfy

$$\int_M \tau_t(\mathbf{r},\mathbf{r}')\,\phi_t(\mathbf{r}',\boldsymbol{\xi})\,\mathrm{d}V' = \lambda_t(\boldsymbol{\xi})\,\phi_t(\mathbf{r},\boldsymbol{\xi}). \tag{143}$$

These functions can be used to define an integral transform in terms of which $\rho_{t-1}(\mathbf{r}')$ will be represented by a function $\sigma_{t-1}(\boldsymbol{\xi})$ such that

$$\rho_{t-1}(\mathbf{r}') = \int \sigma_{t-1}(\boldsymbol{\xi})\,\phi_t(\mathbf{r}',\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}, \tag{144}$$

the integral being taken over the entire range of $\boldsymbol{\xi}$. The law of formation then gives an integral form for $\rho_t(\mathbf{r})$:

$$\rho_t(\mathbf{r}) = \int \lambda_t(\boldsymbol{\xi})\,\sigma_{t-1}(\boldsymbol{\xi})\,\phi_t(\mathbf{r},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}, \tag{145}$$

or, since the functions $\phi_t(\mathbf{r},\boldsymbol{\xi})$ are orthogonal for different $\boldsymbol{\xi}$,

$$\sigma_t(\boldsymbol{\xi}) = \lambda_t(\boldsymbol{\xi})\,\sigma_{t-1}(\boldsymbol{\xi}). \tag{146}$$

If the characteristic functions are independent of t and are properly normalized, the distribution after t steps from a starting point $\mathbf{r}_0$ is

$$\rho_t(\mathbf{r}) = \int \Lambda_t(\boldsymbol{\xi})\,\phi(\mathbf{r}_0,\boldsymbol{\xi})\,\phi(\mathbf{r},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}, \tag{147}$$

where

$$\Lambda_t(\boldsymbol{\xi}) = \prod_{k=1}^{t} \lambda_k(\boldsymbol{\xi}). \tag{148}$$

In the case of a Euclidean space, the characteristic functions are simply $\exp \mathrm{i}(\boldsymbol{\xi}\,.\,\mathbf{r})$ and (144) reduces to a multiple Fourier transform. Thus, we recover the well-known method due to Markoff (1912) (see, for example, Chandrasekhar 1943, p. 8).

There is little formal modification necessary in §§ 5·2 to 5·5 and it may be anticipated that the results of these sections will hold for open manifolds and, in particular, that the class of commutative spaces will include that of harmonic spaces and their products but may be more general.

Appendix A

Let $\{N_K\}$ be any given increasing sequence of integers. We shall define sequences $\{n_k\}$, $\{\alpha_k\}$, $\{\beta_k\}$ such that

$$\{n_k\} \text{ is a subsequence of } \{N_K\}, \tag{A 1}$$

$$0 < \beta_{k+1} < \alpha_k < \beta_k, \tag{A 2}$$

$$\beta_k < \tfrac{1}{2}\pi, \quad \log n_k > 4, \tag{A 3}$$

$$P_{n_k}(\cos\alpha) \geqslant \tfrac{1}{2} \quad \text{for all } \alpha \text{ in } 0 \leqslant \alpha \leqslant \alpha_k, \tag{A 4}$$

$$\sqrt{\frac{2}{n_k \pi \sin\beta_k}} < \frac{1}{\log n_k}. \tag{A 5}$$

We first choose β_1, so that $0 < \beta_1 < \frac{1}{2}\pi$. We then choose n_1 from the sequence $\{N_K\}$ and so large that (A 5) holds for $k = 1$ and that $\log n_1 > 4$. We then choose α_1 in $0 < \alpha_1 < \beta_1$ and so small that (A 4) holds for $k = 1$.

When $n_\kappa, \alpha_\kappa, \beta_\kappa$ have been chosen for $\kappa < k$ we choose β_k in $0 < \beta_k < \alpha_{k-1}$ and then choose n_k from the sequence $\{N_K\}$ and so large that $n_k > n_{k-1}$ and that (A 5) holds, for this value of k. Finally, we choose α_k in $0 < \alpha_k < \beta_k$ and so small that (A 4) holds.

When $\{n_k, \alpha_k, \beta_k\}$ have been chosen in this way we define

$$p(\alpha_k) = p(\beta_k) = \frac{4}{\log n_k}, \quad p(\tfrac{1}{2}\pi) = 1. \tag{A 6}$$

We then complete the definition of $p(\alpha)$ by linear interpolation between the values

$$\ldots \alpha_k, \beta_k, \alpha_{k-1}, \ldots, \beta_1, \tfrac{1}{2}\pi$$

for which it is given by (A 6). Beyond $\frac{1}{2}\pi$, $p(\alpha)$ will be constant; and $p(0) = 0$. Evidently this function $p(\alpha)$ increases monotonically from 0 to 1 as α increases from 0 to π. Evidently, also, $p(\alpha)$ is continuous in the closed interval $(0, \pi)$. But

$$\begin{aligned}
\lambda_{nk} &= \left(\int_0^{\alpha_k} + \int_{\alpha_k}^{\beta_k} + \int_{\beta_k}^{\frac{1}{2}\pi} + \int_{\frac{1}{2}\pi}^{\pi}\right) P_{n_k}(\cos\alpha)\, \mathrm{d}p(\alpha) \\
&= \left(\int_0^{\alpha_k} + \int_{\beta_k}^{\frac{1}{2}\pi}\right) P_{n_k}(\cos\alpha)\, \mathrm{d}p(\alpha), \quad \text{by (A 6)}, \\
&\geqslant \frac{1}{2}\int_0^{\alpha_k} \mathrm{d}p(\alpha) - \int_{\beta_k}^{\frac{1}{2}\pi} \sqrt{\left(\frac{2}{n_k \pi \sin\beta_k}\right)} \mathrm{d}p(\alpha), \quad \text{by (A 4) and (30)}, \\
&> \frac{1}{2}\frac{4}{\log n_k} - \frac{1}{\log n_k} = \frac{1}{\log n_k},
\end{aligned}$$

by (A 5) and (A 6). This establishes (37) with the refinement that the n_k form a subsequence of a sequence $\{N_K\}$ given in advance.

Any such $p(\alpha)$, *without* the refinement, defines a random walk which, applied to the concentrated distribution (17), will never produce convergence. The refinement is needed to obtain a similar result with a bounded ρ_0.

Suppose that ρ_0 is a bounded axially symmetric distribution

$$\rho_0 = \Sigma c_N P_N(\cos\theta),$$

and that

$$c_N > N^\delta \tag{A 7·1}$$

(not for all N but) for an infinity of values of N,

$$N = N_1, N_2, \ldots \tag{A 7·2}$$

with some fixed $\delta > 0$. Then we can find $p(\alpha)$ such that the series for ρ_t will not converge, for any t, at $\theta = 0$.

In constructing ρ_0 we shall include a similar refinement; we suppose an increasing sequence $\{\nu_\kappa\}$ given in advance and shall arrange that

$$\{N_K\} \text{ is a subsequence of } \{\nu_\kappa\}. \tag{A 8}$$

Thus, given either a $p(\alpha)$ satisfying (37) or a ρ_0 satisfying (A 7), we can choose the other in such a way that the series for ρ_t never converges at $\theta = 0$. Moreover, if we are given a $p(\alpha)$

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satisfying (37) we can, by superposing such distributions ρ_0 with different axes, obtain a bounded distribution ρ_0 such that ρ_t diverges for every t at every one of any given finite or enumerably infinite set of points on the sphere.

We need, not a bound for P_n as in (30), but an approximate form for it. This may be obtained by the same deformation of the contour of integration as used for (30): it is

$$P_n(\cos\theta) = \sqrt{\left[\frac{2}{n\pi\sin\theta}\right]}\left\{\cos\{\tfrac{1}{4}\pi - (n+\tfrac{1}{2})\,\theta\} + O\left(\frac{1}{n\sin\theta}\right)\right\}. \qquad \text{(A 9)}$$

It shows that P_n is precisely of the order given by (30) for most α, namely except where the cosine is small.

Given $\{\nu_\kappa\}$, we shall choose, for each integer K, an integer N_K taken from the sequence $\{\nu_\kappa\}$, and an interval (α_K, β_K) satisfying (A 2) and with $\beta_1 < \frac{1}{2}\pi$ as before. We shall put $\rho_0 = 0$ outside these intervals. In (α_K, β_K) we shall put $\rho_0 = 1$ or $\rho_0 = 0$ according as

$$\cos\{\tfrac{1}{4}\pi - (N_K + \tfrac{1}{2})\,\alpha\}$$

is positive or not. And we take $\quad \alpha_K = N_K^{-\epsilon}, \quad \beta_K = 2\alpha_K,$

where ϵ is a fixed positive number less than $\frac{1}{3}$. Then the contribution of the interval (α_K, β_K) to

$$c_{N_K} = (N_K + \tfrac{1}{2}) \int_0^\pi \rho_0 P_{N_K}(\cos\theta)\sin\theta\,\mathrm{d}\theta$$

is

$$\approx \frac{8-\sqrt{8}}{3\pi\sqrt{\pi}}\, N_K^{\frac{1}{2}(1-3\epsilon)}. \qquad \text{(A 10)}$$

We arrange that the other contributions to c_{N_K} are relatively small, and therefore leave c_{N_K} itself approximately equal to (A 10), merely by making the N_K increase sufficiently fast. Because the mean value of the cosine in (A 9) is zero, the contribution of (β_K, π) to c_{N_K} is

$$O(N_K^{-\frac{1}{2}}\, N_{K-1}^{1-\frac{3}{2}\epsilon}),$$

while the contribution of $(0, \alpha_K)$ to c_{N_K} is

$$O(N_K N_{K+1}^{-2\epsilon}).$$

The desired end is achieved by making $N_{K+1} \geqslant N_K^c$ for all large K, where the constant c is chosen so that

$$c > 1, \quad c > \frac{3}{4} + \frac{1}{4\epsilon},$$

of which the second implies the first. From (A 10) we see now that we can satisfy (A 7), with a bounded ρ_0, for any $\delta < \frac{1}{2}$.

Evidently we can achieve a like result with a distribution ρ_0 continuous at $\theta = 0$ (where its value will be zero) and possessing as many derivatives elsewhere as we choose to require. And since our ρ_0 is non-negative we can turn it into a probability distribution by applying a suitable constant factor.

Appendix B

In this appendix we derive properties of a space in which random walks commute. As a first step, we shall endeavour to expand

$$\overline{\rho(\mathbf{r};\alpha)} = \frac{1}{\Omega}\int_{S(\mathbf{r},\alpha)} \frac{\rho(\mathbf{r}')\,\mathrm{d}S'}{\Phi(\mathbf{r},\mathbf{r}')} = \frac{1}{\Omega}\int_{S(\mathbf{r},\alpha)} \rho(\mathbf{r}')\,\mathrm{d}\omega \qquad \text{(B 1)}$$

in a power series in α whose coefficients are functions of $\mathbf{r}$.

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Set up an *affine normal co-ordinate system* y^i with $\mathbf{r}$ as base-point (cf. Veblen 1927, chap. 6). Then the geodesics through $\mathbf{r}$ may be written

$$y^i = \theta^i \alpha, \tag{B 2}$$

where θ^i is the unit contravariant tangent vector at $\mathbf{r}$ and α is the arc length measured from $\mathbf{r}$. Expand $\rho(\mathbf{r}')$ as a Taylor series in y^i, the coefficients being the *affine extensions* of $\rho(\mathbf{r})$ at $\mathbf{r}$ (cf. Ruse 1930*b*)

$$\rho(\mathbf{r}') = \rho(\mathbf{r}) + y^i \rho_i(\mathbf{r}) + \frac{1}{2!} y^i y^j \rho_{ij}(\mathbf{r}) + \dots. \tag{B 3}$$

It will be convenient temporarily to specialize the co-ordinate system still further to secure that

$$g_{ij}(\mathbf{r}) = g^{ij}(\mathbf{r}) = \delta_{ij}, \quad \theta^i = \theta_i. \tag{B 4}$$

The θ^i are then direction cosines and satisfy

$$\theta_1^2 + \theta_2^2 + \dots + \theta_q^2 = 1. \tag{B 5}$$

This choice of co-ordinates frees superfixes for use as exponents in the following analysis: the co-ordinates may be called *pseudo-Euclidean* co-ordinates with $\mathbf{r}$ as base point.

Denote by $M_{n_1 n_2 \dots n_q}$ the average value of $\theta_1^{n_1} \theta_2^{n_2} \dots \theta_q^{n_q}$ over all possible directions,

$$M_{n_1 n_2 \dots n_q} = \frac{1}{\Omega} \int_U \theta_1^{n_1} \theta_2^{n_2} \dots \theta_q^{n_q} \, \mathrm{d}\omega, \tag{B 6}$$

where U denotes the unit-sphere (B 5) and $\mathrm{d}\omega$ is the element of q dimensional angle. It is clear that $M_{n_1 n_2 \dots n_q}$ vanishes if any n_i is odd: let us write

$$n_i = 2\nu_i, \quad \sum_i \nu_i = \nu, \quad n = 2\nu = \sum_i n_i, \tag{B 7}$$

and we expect to be concerned only with integral values of ν_i, ν.

To evaluate the quantities M we use a generating-function method. Consider the integral

$$I(\xi) = I = \int_Y \exp\left[\sum_i \xi_i y_i - \sum_i y_i^2\right] \mathrm{d}y_1 \, \mathrm{d}y_2 \dots \mathrm{d}y_q \tag{B 8}$$

taken over the space Y of all real values of the y_i. Expanding the integrand in powers of the ξ_i we get

$$\begin{aligned} I &= \sum_{(ni)} \left(\Pi \frac{\xi_i^{n_i}}{n_i!} \right) \int_Y \mathrm{e}^{-\Sigma y_i^2} \Pi y_i^{n_i} \, \mathrm{d}y_i, \\ &= \sum_{(ni)} \left(\Pi \frac{\xi_i^{n_i}}{n_i!} \right) M_{n_1 n_2 \dots n_q} \Omega \int_0^\infty \mathrm{e}^{-\alpha^2} \alpha^{n+q-1} \, \mathrm{d}\alpha, \\ &= \sum_{(ni)} \left(\Pi \frac{\xi_i^{n_i}}{n_i!} \right) M_{n_1 n_2 \dots n_q} \pi^{\frac{1}{2}q} \Gamma\left(\frac{n+q}{2}\right) \Big/ \Gamma\left(\frac{q}{2}\right). \end{aligned} \tag{B 9}$$

But, by a change of origin in (B 8), we find

$$I = \pi^{\frac{1}{2}q} \, \mathrm{e}^{\frac{1}{4}\Sigma \xi_i^2}.$$

Equating coefficients here and in (B 9) we see that $M_{n_1 n_2 \dots n_q}$ vanishes unless the n_i are all even, when, with the convention (B 7),

$$M_{n_1 n_2 \dots n_q} = \frac{4^{-\nu} \Gamma(\frac{1}{2}q)}{\Gamma(\nu + \frac{1}{2}q)} \Pi \frac{n_i!}{\nu_i!}. \tag{B 10}$$

Now the expansion (B 3) inserted in (B 1) gives us

$$\overline{\rho(\mathbf{r};\alpha)} = \sum_{(n_i)} \alpha^n M_{n_1 n_2 \dots n_q} \left(\Pi \frac{\partial_i^{n_i}}{n_i!}\right) \rho(\mathbf{r}), \tag{B 11}$$

where ∂_i denotes $\partial/\partial y_i$. Substituting from (B 10) we get

$$\overline{\rho(\mathbf{r};\alpha)} = \sum_{\nu=0}^{\infty} \frac{\Gamma(\frac{1}{2}q)}{\nu!\,\Gamma(\nu+\frac{1}{2}q)} \left(\tfrac{1}{4}\alpha^2 \sum_i \partial_i^2\right)^\nu \rho(\mathbf{r}), \tag{B 12}$$

$$= \sum_{\nu=0}^{\infty} \frac{\Gamma(\frac{1}{2}q)}{\nu!\,\Gamma(\nu+\frac{1}{2}q)} \left(\tfrac{1}{4}\alpha^2\right)^\nu \Delta_{2\nu}\rho(\mathbf{r}), \tag{B 13}$$

say. For $\nu = 1$ this notation agrees with the accepted notation Δ_2 for the Beltrami differential operator, denoted by ∇^2 in our work. Returning to a general affine normal co-ordinate system with $\mathbf{r}$ as base-point, i.e. abandoning the special restrictions (B 4)

$$\Delta_{2\nu}\rho(\mathbf{r}) = g^{i_1 j_1} g^{i_2 j_2} \dots g^{i_\nu j_\nu} \rho_{i_1 j_1 i_2 j_2 \dots i_\nu j_\nu}(\mathbf{r}), \tag{B 14}$$

which expresses $\Delta_{2\nu}\rho$ as a contraction of an affine extension of ρ.

The operator $\Delta_{2\nu}$ is, as (B 12) shows, in a certain sense a power of Δ_2, but not in the strict sense. In fact, if we define

$$\Delta_{2;\mathbf{r}} \equiv g^{ij}(\mathbf{r}) \frac{\partial^2}{\partial y^i \partial y^j},$$

where it is understood that y^i, y^j are affine normal co-ordinates, with $\mathbf{r}$ as base-point, of a general point $\mathbf{r}'$ in some neighbourhood of $\mathbf{r}$, then

$$\Delta_{2\nu}\rho(\mathbf{r}) = [\Delta_{2;\mathbf{r}}^{\nu}\, \rho(\mathbf{r}')]_{\mathbf{r}'=\mathbf{r}}. \tag{B 15}$$

From (B 13) we can immediately write down the effect of a random step of length α followed by a random step of length β. If we write

$$D_{q,\nu} = \frac{\Gamma(\frac{1}{2}q)}{\nu!\,\Gamma(\nu+\frac{1}{2}q)} 4^{-\nu},$$

it is

$$\overline{\rho(\mathbf{r};\alpha,\beta)} = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} D_{q,\mu}\beta^{2\mu} D_{q,\nu}\alpha^{2\nu} \Delta_{2\mu}\Delta_{2\nu}\rho(\mathbf{r}). \tag{B 16}$$

The condition that these random steps commute, for all values of α and β, is that

$$\Delta_{2\mu}\Delta_{2\nu}\rho(\mathbf{r}) = \Delta_{2\nu}\Delta_{2\mu}\rho(\mathbf{r}) \tag{B 17}$$

for all μ and ν: and if this is to hold for all distributions ρ then

the operators $\Delta_{2\nu}$ commute. (B 18)

The above analysis for random steps of fixed length α generalizes at once to problems with the third extension, the power $\alpha^{2\nu}$ being replaced by the 2νth moment

$$\overline{\alpha^{2\nu}} = \int \alpha^{2\nu}\,\mathrm{d}p(\alpha)$$

of the distribution of α. The change has no effect on the argument leading to (B 18) as the condition for commutability. Our analysis assumes $\rho(\mathbf{r})$ analytic: but this restriction can be removed. But it also assumes the *manifold* of class C^ω, i.e. the metric g_{ij} analytic. If this is not given, the conditions (B 18) are necessary for commutability, so far as they have sense, but it is not proved that they are sufficient.

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Appendix C

We suppose given an analytic closed Riemannian manifold. The whole manifold is covered by a finite number of regions: each of these regions possesses a local co-ordinate system: the transformation from one local co-ordinate system to another local co-ordinate system is analytic in the overlap in which both systems apply: and the components of the metric tensor g_{ij} in any one of these co-ordinate systems are analytic functions of those co-ordinates in the region in which those co-ordinates apply. Only analytic transformations of co-ordinates are allowable.

We suppose also that the metric is such that equation (127) holds everywhere, i.e. equation (B 17) holds at any point $\mathbf{r}$ for any function of position ρ defined in the neighbourhood of $\mathbf{r}$ and differentiable at $\mathbf{r}$ the relevant number of times.

We are to prove that any two random steps commute. This is to apply for any 'distribution' in our sense, i.e. for any bounded additive function of Borelian sets E, not merely for those given by the integral $\int_E \rho \, \mathrm{d}V$ of a true density ρ.

The first stage is to prove the result locally for locally analytic densities.

The second stage removes the restriction to analytic densities, but the lengths α, β of the two steps will still be restricted to values sufficiently small.

The final stage removes this restriction to small values of α and β.

Consider then a point $\mathbf{r}_0$ of the manifold and set up pseudo-Euclidean co-ordinates (y_j) with this point as origin. A region $|y| < \delta$ in the Cartesian space Y of co-ordinates (y_j) is mapped (1-1) analytically on a certain neighbourhood of $\mathbf{r}_0$.

Any point y in Y determines a corresponding point $\mathbf{r}_y$ on the manifold, and whatever co-ordinate system (x_i) be allowable in the neighbourhood of $\mathbf{r}_y$, those co-ordinates x_i are analytic functions of the y_j, but there may be points at which the y_j are not analytic functions of the x_i, since the Jacobian $\partial(x_i)/\partial(y_j)$ may vanish.

Corresponding to the pseudo-Euclidean co-ordinates y_j we have a set of q orthonormal tangent vectors e_j at $\mathbf{r}_0$. Parallel transport of these along the geodesic represented in Y by the radius from 0 to y will produce a set of q orthonormal tangent vectors e_{jy} at $\mathbf{r}_y$. We use these vectors e_{jy} at $\mathbf{r}_y$ as the unit vectors of a set of pseudo-Euclidean co-ordinates (z_k) with the point $\mathbf{r}_y$ as origin. As before, there is a region $|z| < \delta$ in the Cartesian space Z of co-ordinates (z_k) which is mapped (1-1) analytically on a certain neighbourhood of $\mathbf{r}_y$. *Any* point z in Z determines a corresponding point $\mathbf{r}_{yz}$ on the manifold. Whatever co-ordinate system (x_i) be allowable in the neighbourhood of $\mathbf{r}_{yz}$, those co-ordinates x_i are analytic functions of the $2q$ variables

$$y_1, \ldots, y_q; \; z_1, \ldots, z_q.$$

To be analytic in (y, z) at (y', z') is to be equal, in some neighbourhood of (y', z'), to a multiple Taylor series about (y', z'). The statement that the co-ordinates of $\mathbf{r}_{yz}$ are analytic functions of the $2q$ variables (y_j, z_k) may be proved, without going outside the real field, by a discussion of the relevant power series, their convergence, and the effect of transforming to a different set of local co-ordinates: the proof on such lines would be cumbersome. A different proof is obtained if we consider the given manifold as embedded in one in which the allowable co-ordinates x_i, and hence also the y_j and z_k, are allowed to take complex values, albeit confined to small values of the imaginary part. The metric components g_{ij} lose their metric significance in the wider field, and will in general assume complex values. It is sufficient to prove that the first derivatives $\partial x_i/\partial y_j$, $\partial x_i/\partial z_k$ exist in this wider field in order to infer that x_i is an analytic function of the $2q$ complex variables y_j, z_k.

The density $\overline{\rho(\mathbf{r}_0;\,\alpha,\beta)}$ produced at $\mathbf{r}_0$ by two random steps of lengths α, β is simply the average of $\rho(\mathbf{r}_{yz})$ over the locus given by

$$|y|^2 = y_1^2+\ldots+y_q^2 = \beta^2, \tag{C 1}$$

$$|z|^2 = z_1^2+\ldots+z_q^2 = \alpha^2. \tag{C 2}$$

If $\rho(\mathbf{r})$ is analytic at $\mathbf{r}_0$, then $\rho(\mathbf{r}_{yz})$ is analytic in the variables (y_j, z_k) at $(0, 0)$. For α and β sufficiently small, it is expressible as a multiple power series in the y_j, z_k. The standard theorems on integration of uniformly convergent series enable us to integrate over (C 2) term by term: this gives (B 13) with $\mathbf{r}_y$ for $\mathbf{r}$, and the assurance that the individual terms are themselves uniformly convergent power series in the y_j. Finally, integration term by term over the locus (C 1) gives the double power series (B 16), with the assurance that it does converge for sufficiently small α, β. Thus

$$\overline{\rho(\mathbf{r}_0;\,\alpha,\beta)} = \overline{\rho(\mathbf{r}_0;\,\beta,\alpha)}$$

for any $\rho(\mathbf{r})$ which is analytic at $\mathbf{r}_0$ and for sufficiently small α, β. This completes the first stage. We remark that the bounds applied to α and β arise from two causes, the geometry of the manifold and the nature of $\rho(\mathbf{r})$. Fundamentally we are concerned with substituting

$$x_i = \text{power series in } y_j,\, z_k \tag{C 3}$$

into an expansion

$$\rho = \text{power series in } x_i, \tag{C 4}$$

where now for the moment the x_i are allowable co-ordinates with $\mathbf{r}_0$ as origin. The series in (C 3) is dominated by some $X_i(\alpha, \beta)$, i.e. the sum of the moduli of its terms is less than $X_i(\alpha,\beta)$; and α, β must be taken so small that not only are the X_i finite but that the series in (C 4) is absolutely convergent, and equal to ρ, for all (x_i) satisfying $|x_i| < X_i$.

In the second stage we apply the above with locally analytic densities ρ which 'approximate' to a Dirac δ-function $\delta(\mathbf{r}, \mathbf{r}_1)$. It is sufficient to consider

$$\rho = A\,\mathrm{e}^{-Bs^2},$$

where s is the shortest geodesic distance from $\mathbf{r}_1$, B tends to infinity, and the constant A is so adjusted that the integral of ρ over the region in which it differs appreciably from zero shall be approximately unity, i.e.

$$A \sim (B/\pi)^{\frac{1}{2}q}, \quad B \to \infty.$$

We infer commutativity for steps of lengths α, β sufficiently small as applied to an initial distribution given by a Dirac δ-function. The bound that remains on α, β depends simply on the geometry of the manifold. The restriction previously depending on the analytic character of ρ now depends simply on that of s^2.

Commutativity, subject to these restrictions on α and β, follows for an arbitrary initial distribution from commutativity for an initial δ-distribution by the obvious device of superposition. This completes the second stage.

Consider now the distributions $\rho'(\mathbf{r})$, $\rho''(\mathbf{r})$ formed from an initial δ-distribution at $\mathbf{r}_0$ by a step of length β followed by one of length α and by a step of length α followed by one of length β. The former is obtained by superposing

$$\text{charge } \mathrm{d}\omega_y \mathrm{d}\omega_z/\Omega^2 \quad \text{at} \quad \mathbf{r}_{yz}$$

for all pairs of elementary solid angles $\mathrm{d}\omega_y$, $\mathrm{d}\omega_z$ on the loci (C 1), (C 2). Thus

$$\int_M \rho'(\mathbf{r})\,\phi_n(\mathbf{r})\,\mathrm{d}V = \iint \phi_n(\mathbf{r}_{yz})\,\mathrm{d}\omega_y\,\mathrm{d}\omega_z/\Omega^2. \tag{C 5}$$

Since $\mathbf{r}_{yz}$ is analytic in y_j, z_k and since $\phi_n(\mathbf{r})$ is analytic (the manifold being analytic), it follows that the right-hand member of (C 5) is analytic in α, β. So also therefore is the left-hand member: and for like reason

$$\int_M \rho''(\mathbf{r})\,\phi_n(\mathbf{r})\,\mathrm{d}V \tag{C 6}$$

is analytic in α, β. Since (C 5) and (C 6) are both analytic in α and β, and identically equal for small (α, β), it follows that they are equal for all α, β. Since this is true for each n, and since the ϕ_n are a complete system, it follows that

$$\rho'(\mathbf{r}) = \rho''(\mathbf{r}).$$

And since steps of arbitrary length commute when applied to an initial δ-distribution, it follows by superposition that they commute for an arbitrary initial distribution.

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